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BROADBAND MATCHING BETWEEN ARBITRARY
LOAD AND SOURCE IMPEDANCES

Prepared by

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PREFACE

The report contained herein is identical with a thesis of the same title presented to the faculty of the Graduate Division of Georgia Institute of Technology in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Electrical Engineering.

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ABSTRACT

Broadband matching between arbitrary load and source impedances is investigated. An ideal impedance match over a finite frequency range is interpreted herein as a constant power transfer over the frequency range, the constant power being equal to the maximum power available from the generator. Except for the case of very simple load and source networks, it is not possible to achieve the ideal matching condition. Neither is it generally possible to transfer a constant, though less than maximum, power over a finite frequency range with a finite number of elements in the matching section. It is possible, however, to transfer a "reasonably constant" power over a finite frequency range with matching sections having a finite number of elements. The allusion to "reasonably constant" power transfer indicates that the power transfer level is confined between an upper and lower limit in the pass band. The tolerance or ripple is a measure of the difference between the upper and lower limits. For each particular method of specifying the form of behavior in the pass band, there exists an interdependence between the level of power transfer and the tolerance. For convenience, the power level is expressed herein in terms of network transmission and reflection coefficients and return loss functions.

The solution to the problem utilizes a cascade connection of a load impedance, a lossless matching network, and a lossless source network interposed between a purely resistive generator impedance and

the matching network. The load impedance and source network, though arbitrary, are known. Through the Darlington equivalent circuit representation, the load impedance is specified as a lossless two terminal-pair (known as the load network), terminated in a one-ohm resistance. Previous work, notably by Fano and Bode, deals with cascaded load impedances and matching networks, wherein the characteristics of the overall system are investigated through constraints imposed by the load network only. The additional specification of the source networks of this study introduces constraints at both ends of the overall system. It is because of these constraints that the problem of this study differs from previous work.

The objectives considered in the study are listed below.

(a) Determine in terms of the reflection and transmission coefficients of the source and load networks the conditions of physical realizability and limits of bandwidth and matching level imposed on the overall system (source, matching, and load networks) by the known load and source networks.

(b) Develop network functions which are in accord with the realizability conditions and which lead to realizable matching networks with a finite number of elements.

(c) Investigate methods and procedures for applying the results of (a) and (b) to systems having load and source networks of high complexity.

(d) Develop methods of constructing matching networks of any desired complexity consistent with (a) and (b) above.

(e) Demonstrate the methods of (d) by several examples.

The problem of this study is solved within the framework of specific types of parameters. For example, all the networks, i.e., load, matching, source, or any combination thereof, are lossless two-terminal-pair networks. The terminations are normalized as one-ohm resistances. The load impedance becomes the load network and terminating resistance through use of a Darlington equivalent network. Modern network theory is employed for describing the behavior of lossless networks terminated in one-ohm resistances.

Through use of Tchebycheff functions of the first kind, a form of matching behavior is preselected which leads to variation of power transfer between fixed limits in the pass band and a monotonic approach to a final value in the stop band.

Subsequent to the choice and introduction of the Tchebycheff form of behavior, emphasis of the matching problem is centered on low pass transmission, with the real-frequency transmission zeros located at infinity. In this type of transmission, matching takes place over a band of frequencies extending from zero to some finite value of frequency.

The necessary and sufficient conditions of physical realizability imposed on the overall system are found in terms of the reflection and transmission coefficients of the known load and source networks. As a consequence, Taylor series expansions of the return losses about transmission zeros of the known load and source networks can be made. The coefficients of the resultant expansions contain information about conditions which the zeros and poles of the overall reflection coefficients must satisfy.

A set of contour integrals serves to establish the link between the behavior of the overall reflection coefficient at real frequencies and the coefficients of the Taylor series expansions of return losses about transmission zeros. The integrands consist of the return losses multiplied by weighting functions which introduce poles of appropriate multiplicity at the zeros of transmission. Two sets of equations result when the products are integrated around a closed contour in the s -plane. Through appropriate manipulation, the integrals serve to specify the behavior at real frequencies of the magnitude of a prescribed return loss in terms of the Taylor series coefficients.

In order to obtain a consistent set of equations for relating the integrals to the Taylor series coefficients, it often is necessary to introduce additional terms which appear herein as simple right-half s -plane reflection zeros or double zeros of reflection on the positive real axis. In particular, the added double zeros of reflection release a condition of fixed difference between corresponding pairs of integral equations. It is to be noted, however, that the addition of either type of reflection zero always has the effect of reducing the possible gain.

Certain conclusions are reached with regard to gain and tolerance, where gain is used herein as the level of power transfer in the pass band and can be expressed in terms of $|t|$ or $\ln(1/|\rho_1|)$ and tolerance is a measure of the ripple or variation in gain throughout the pass band.

For particular load and source networks and a fixed n , it is possible to (a) obtain any desired gain less than a fixed limiting

value if no restriction is placed on the tolerance and (b) to obtain any desired tolerance less than a fixed limiting value if no restriction is placed on gain. In regard to (a) and (b) above, a decrease in tolerance always causes a decrease in gain.

If both a desired gain and tolerance are specified in advance, it may or may not be possible to find a suitable matching network. In the event it is not possible, a compromise solution must be made.

CHAPTER I

INTRODUCTION

Definition of the Problem

Power transfer.--At a fixed real frequency the problem of obtaining maximum power transfer between an energy source and an unrestricted load is solved easily by making the load impedance the complex conjugate of the source impedance. On the other hand, if the load is restricted and does not permit conjugacy, the power transferred is usually less than that of the unrestricted case, but never greater than that of the unrestricted case.

In variable frequency circuits the discussion of maximum power transfer becomes more complicated than in fixed frequency circuits, since the parameters involved usually are functions of frequency. A typical variable frequency problem can be represented by the attempt to obtain a power match over a finite frequency band between a generator having a purely resistive internal impedance and a complex load impedance Z_L . If the conditions of the problem postulate a lossless two-terminal-pair matching section, it can be shown that a perfect match cannot always be obtained for all frequencies in a finite band, for such a requirement would demand that the matching network present a complex conjugate impedance to the generator over a finite band of frequencies. Furthermore, an infinite number of elements and complete freedom of choice of those elements are required in the matching section if the power match (less than perfect though it might be) is to be

constant over a finite band of frequencies (1).

As restrictions are placed on the source and load, the ability of the matching section to effect even a reasonably constant match is altered drastically. If, in addition to a load Z_L , a specified source impedance is interposed between the generator and the matching section, the added restrictions and conditions of matching are such as to warrant a careful and complete examination of the theoretical limitations and physical realizability conditions imposed on the complete network. This study is devoted to the investigation of the limitations and physical realizability conditions imposed on systems utilizing lossless networks between arbitrary generator source impedances and arbitrary load impedances.

Matching and filter sections.--Prior to formulating a concise statement of the problem with definite objectives and methods, it is convenient to discuss briefly some of the general properties of matching and matching sections. A more detailed description of the matching sections and their relationship to the source and load sections is presented in Chapter II, under the heading "The Networks."

Two-terminal-pair networks of the type considered have wide application as filter networks as well as matching networks. In many respects, matching and filter networks are similar in that (a) transmission is desired over certain frequency ranges and not desired over others, (b) the methods of synthesis are often identical, and (c) the final form and physical arrangement of elements are often similar. In other respects, however, filter networks and matching networks can differ markedly. Filters are often designed to have a perfect impedance

match at several frequencies in a pass band, although at other frequencies within the pass band the impedance match may be poor. The poorer impedance match at some frequencies is accepted in exchange for benefits of sharper cut-off characteristics and good attenuation characteristics in the stop band. On the other hand, matching networks are usually found to transfer power inefficiently when perfect impedance matches exist at several frequencies. The criterion for matching in this study is optimum power transfer, constant within given limits, over a specified frequency range. For convenience, the absolute value of the complex return loss is used as a measure of power transfer. The complex return loss is the principal value of the natural logarithm of the reciprocal of the reflection coefficient. A discussion of the criterion for optimum power transfer is contained in Chapter III.

Statement of the problem.--In this study, a two-terminal-pair source network and a load impedance are specified. A suitable two-terminal-pair matching network is to be determined and interposed between the source network and the load impedance. In order to utilize the properties of lossless networks, the load impedance is replaced by its Darlington equivalent network (2), which consists of a lossless two-terminal-pair network terminated in a pure resistance. The input impedance of the Darlington equivalent network is identical with that of the load impedance which it replaces. The introduction of the Darlington equivalent network completes a set of cascaded, lossless two terminal-pairs consisting, in order, of the source network, the matching network, and the load network. The load network is the lossless part of the Darlington equivalent of the load impedance. The above

combination, which can be considered as a lossless two terminal-pair in itself, is terminated on the source end by the generator resistance and on the load end by the Darlington network resistance. Concise statements of the specific objectives of this study are as follows:

(a) Determine in terms of the reflection and transmission coefficients of the source and load networks the conditions of physical realizability and the limits of bandwidth and matching level imposed on the overall system of source, matching, and load networks by the known load and source networks.

(b) Develop network functions which are in accord with the realizability conditions and which also lead to realizable matching networks with a finite number of elements.

(c) Investigate methods and procedures for applying the results of (a) and (b) to load and source networks of high complexity.

(d) Develop methods to construct matching networks of any desired complexity, consistent with the conditions of (a) and (b) above.

(e) Demonstrate the methods of (d) by means of examples.

Reflection and transmission coefficients are used extensively herein to describe the behavior of the network functions. They are functions of the complex frequency variable $s = \sigma + j\omega$ and are discussed in detail in Chapter II. As was mentioned earlier, the complex return loss is directly related to the reflection coefficients. The necessary and sufficient conditions for physical realizability of the overall system (in which the matching network is included) are given in the following paragraph for convenience in discussing the more general aspects of this study. These conditions are developed in Chapter III.

All the transmission zeros (i.e., zeros of the transmission coefficients) of the source and load networks which lie in the right-half s -plane or on the imaginary axis must occur as transmission zeros of the complete system with at least the same multiplicity. In addition, a certain number of coefficients of the Taylor series expansions about transmission zeros for the return losses of the source and load network must be equal to the corresponding coefficients of return loss expansions of the complete system (source, matching, and load networks).

A set of contour integrals serves to establish the link between the behavior of the reflection coefficients at real frequencies (i.e., along the imaginary axis in the s -plane) and the coefficients of the Taylor series. To show this, the return losses are multiplied by weighting functions which introduce poles of appropriate multiplicity at the zeros of transmission. Two sets of equations result when the return loss weighting function products are integrated about a closed contour in the s -plane. The integration contour consists of the imaginary axis and the semicircle at infinity in the right-half s -plane.

Through use of an approximating function which specifies certain relationships among the poles and zeros of the transmission and reflection coefficients, it is possible to assemble a set of equations from the integral relationships and thereby determine the reflection coefficients of the complete system as a ratio of polynomials in s . Once the expressions for the reflection coefficients of the complete system have been found, it is a relatively simple matter to obtain the corresponding input impedances and synthesize the matching network.

It is shown in Chapter III that in case the integral relation-

ships do not at first yield a consistent set of equations, it is possible to introduce terms which serve to bring about consistency. The additive terms are in the form of right-half s-plane zeros of the reflection coefficients. Whenever additional right-half s-plane reflection coefficient zeros are introduced, a reduction in possible gain always results.

The major differences between the problem considered herein and previous work arise at this point. In previous work, only the load is specified, leaving complete freedom of choice of the elements at one end (source end) of the matching network. Because of this freedom of choice, an optimizing procedure can be developed which results in finite networks with least maximum absolute values of reflection coefficients. If the resultant tolerance or "ripple" in reflection coefficient is not sufficiently small, the alternative is to redesign the network for an increased number of elements.

In the problem of this study, necessary and sufficient conditions for physical realizability are developed simultaneously from behavior at the load and source ends of the network system rather than at just one end. It is found that the development of these conditions and the associated integral relationships not only describe behavior at each end of the network system, but also reveal that the integral relationships developed at opposite ends of the network system are not independent of each other. It is shown in Chapter III that for a given type of response (Tchebycheff, Butterworth, etc.) and a given complexity of matching section, the specification of both the load and source networks leads to a unique matching network which may or may not be

within acceptable limits of tolerance. Furthermore, the tolerance does not necessarily decrease to zero as the number of elements is increased indefinitely. As shown in Chapter III, this disturbing situation is especially evident when the element of the load network nearest the terminating resistance is equal to the element of the source network nearest the generator resistance. In this case, there exists the undesirable (undesirable from a matching standpoint) condition of perfect match at some frequencies and poor match at others. The situation is not improved appreciably as the complexity of the matching network increases.

The difficulty reported above is alleviated by imbedding an all-pass structure within the matching network. The effect of imbedding the all-pass structure is to introduce a double right-half s-plane zero of reflection, which adds a term to one set of contour integral relations and does not alter the other set. By proper selection of the added term, the tolerance can be controlled. The introduction of the added term and the means for inserting its effects into the network equations are considered as a contribution of this study.

History and Background

Bode's contribution.--Bode (3) investigates the problem of finding a lossless section to match a parallel resistance-capacitance load to a generator source resistance. Regardless of what matching conditions are used, the system of the matching network and R-C load must satisfy the inequality

$$\int_0^{\infty} \ln \frac{1}{|e|} d\omega \leq \frac{\pi}{RC} , \quad (1)$$

where $|e|$ is the absolute value of the reflection coefficient of the system at real frequencies, and R and C are the load parameters. Because of the limit on the integral of equation (1), any attempt to make $|e|$ small at any frequency in a pass band has the effect of either reducing the bandwidth or causing $|e|$ to be large somewhere else in the pass band. If $\ln(1/|e|)$ is constant at a value $\ln(1/|e|)_{\max}$ over a bandwidth of ω_c radians per second and is zero elsewhere, the result of the integration is

$$\ln \frac{1}{|e|_{\max}} \leq \frac{\pi}{\omega_c RC} . \quad (2)$$

It is seen that the maximum value of $\ln(1/|e|)$ is limited by the value $\pi/\omega_c RC$. Bode (4) shows that when $\ln(1/|e|)$ is constant over the pass band and zero outside the pass band, the most efficient use of the integral of equation (1) is made.

Fano's contribution.--Fano (5) investigates the problem of matching an arbitrary load impedance to a source resistance by means of a lossless two-terminal-pair network. Necessary and sufficient conditions for the physical realizability of input reflection coefficients of matching networks terminated in arbitrary load impedances are developed from coefficients of Taylor series expansions of $\ln(1/e)$ about transmission zeros. The physical realizability conditions are converted to a set of contour integral relationships, one of which is Bode's constraint noted above. The set of contour integral relationships serves to establish the connection between $|e|$ at real frequencies and the Taylor coefficients.

In considering the input reflection coefficient, the arbitrary load impedance is replaced by the equivalent Darlington representation (6) of a physically realizable impedance function as a purely reactive two terminal-pair, terminated in a pure resistance. The complexity of the Darlington equivalent circuit determines the number of necessary and sufficient conditions for physical realizability and thus the number of contour integral relations. The contour integral relations, in turn, fix the optimum bandwidth and tolerance of match.

The optimum bandwidth and tolerance of match can be obtained only when an infinite number of elements are used in the matching section. Fano develops a method of approaching this theoretical limit by means of a matching network with a finite number of elements. Tchebycheff functions are used to determine a reflection coefficient, the absolute value of which oscillates between two given values in the pass band and approaches unity monotonically in the stop band.

Once the reflection coefficient is established, it is possible to find the appropriate matching network through methods of modern network synthesis. Fano develops matching networks for simple but important types of loads and points out the difficulties which might be expected in applying his methods to more complex loads.

Carlin and LaRosa's contribution.--Carlin and LaRosa (7), (8) extend Fano's matching problem in a somewhat different direction. Instead of using lossless matching sections and attempting to obtain constant magnitude of reflection coefficients, lossy matching sections are used and load power replaces $\ln(1/|\rho|)$ as the parameter to be optimized. Carlin and LaRosa employ scattering matrices in conjunction with Fano's

integral relationships to effect a solution which may lead to simpler networks than those derived by Fano's methods. One disadvantage of the Carlin and LaRosa method is that dissipation of energy occurs in the matching section.

Purpose

Practical applications.---The methods described herein can be applied to problems in which a power match between source and load is desired over a given frequency band. The methods are particularly useful when the match is to be made between a load and a network system which can be represented as a source network in series with the resistive impedance of a generator.

One practical example occurs in the study of four-terminal amplifier interstage networks. The plate-to-cathode capacitance forms a source network, while the input circuit to the succeeding vacuum tube stage forms the load. A matching network can be placed between these source and load networks for broadband matching purposes.

Another example occurs when a broadband match is desired between a prescribed source network and an impedance matching transformer. The simplest equivalent circuit of the transformer and resistive termination is a series L-R combination, which can be interpreted as a simple load network and resistive termination.

A third example occurs in the study of matching a transmission line to a given load impedance. If a lossless lumped-parameter equivalent circuit is used to represent the transmission line, it can be interpreted as a source network. The matching network can then be found by application of the methods of this study.

Theoretical considerations.--Aside from any practical applications, the theoretical aspects of the problem are sufficiently important to be an end in themselves. Known results are extended and generalized, and the consideration of the double zero of a reflection coefficient indicates a mode of attack which has not been exploited previously.

CHAPTER II

GENERAL THEORETICAL CONSIDERATIONS

The Networks

Three distinct networks are considered in this study: the load network, the source network, and the matching network. Each network is in the form of a lossless two terminal-pair, although originally the load network might appear in a different form which can be transformed to a lossless two terminal-pair by Darlington's method (9).

The load network.--The load network consists of a lossless two terminal-pair which is terminated on the far end by a load resistance. The load network and termination are derived from the given arbitrary load impedance Z_L by application of the Darlington representation (10) of the load impedance as the input impedance of a lossless two terminal-pair terminated in a resistance.

The source network.--The source network is a two terminal-pair connected in cascade between the generator resistance and the remainder of the circuit, as shown in Fig. 1. Although originally the source network is completely arbitrary except for being lossless, the actual network which is used in the analysis is the Darlington representation of the original source network.

The matching network.--The matching network is a lossless two terminal-pair which is connected in cascade between the load network and the source network.

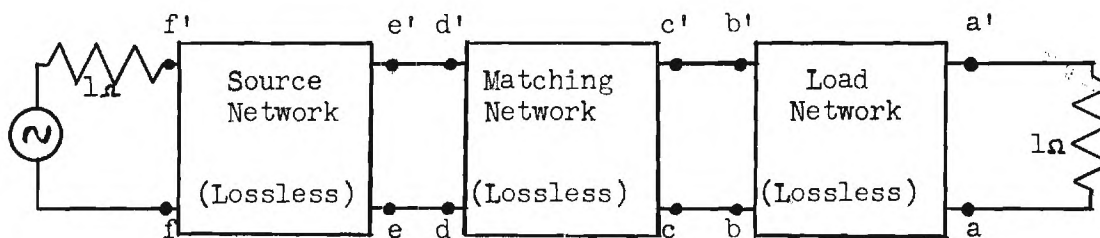


Fig. 1. Source, Matching, and Load Networks

Techniques for Description of the Networks

The load and source networks are derived from previously stipulated networks, whereas the matching network is to be determined. The complete network consists of a generator represented as a source in series with a resistance, the source network, the matching network, the load network, and the load resistance. The arrangement of the networks is shown in Fig. 1.

Complex frequency variable.---The complex frequency variable $s = \sigma + j\omega$ is used in the description of impedances, reflection coefficients, etc., in order to take advantage of applicable mathematical methods from complex variable theory.

Normalization.---The generator and load resistances can be normalized conveniently at values of one ohm each by using, if necessary, ideal transformers. This process permits a description of the lossless networks in terms of one-ohm resistive terminations. Further, when both terminations are one-ohm resistances, it is immaterial which end of the system is considered the source end or the load end. This is true because the principle of reciprocity indicates that power can be transferred equally well in either direction in a lossless network termi-

nated by equal resistances.

Form of behavior.--Transmission in the pass band is specified to be of the "equal ripple" type, while transmission in the attenuation band monotonically approaches zero. This behavior is achieved through use of Tchebycheff polynomials, as is explained in Chapter III.

Analytic Properties of Reflection and Transmission Coefficients

General properties.--Any two-terminal-pair network such as the load, source, or matching network, or any cascade combination of these networks requires three independent circuit parameters to specify the network completely. Two reflection coefficients and the transmission coefficient of the lossless networks terminated in one-ohm resistances can be used as the three necessary parameters. Appendix A describes the development of reflection and transmission coefficients. A reflection coefficient is the ratio of the "reflected" voltage to the "incident" voltage at a pair of terminals, while a transmission coefficient is the ratio of the "transmitted" voltage to the "incident" voltage. The general arrangement of a one-ohm terminated network is shown in Fig. 2. As indicated in Appendix A, the defining equations for the reflection and transmission coefficients are

$$\rho_1 = \frac{2V_1 - E_1}{E_1} \bigg|_{E_2=0}, \quad (3)$$

$$\rho_2 = \frac{2V_2 - E_2}{E_2} \bigg|_{E_1=0}, \quad (4)$$

$$t = \frac{2V_2}{E_1} \bigg|_{E_2=0} = \frac{2V_1}{E_2} \bigg|_{E_1=0}. \quad (5)$$

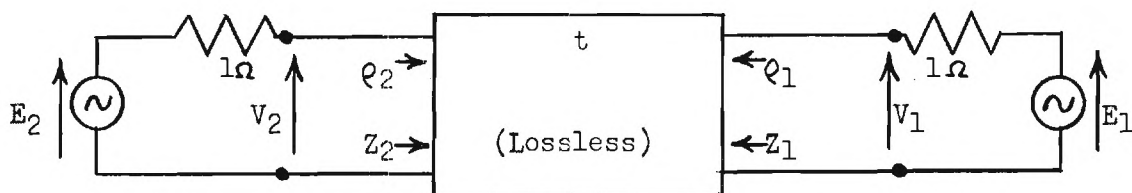


Fig. 2. One-ohm Terminated Network

The parameter ρ_1 is defined as the reflection coefficient of a lossless two terminal-pair terminated in a one-ohm resistance. The input impedance of the two terminal-pair is Z_1 . The parameter ρ_2 has a similar description in terms of an input impedance Z_2 . The parameter t is the transmission coefficient of a lossless two-terminal-pair network terminated on both ends by one-ohm resistances. Reference to Fig. 2 shows a lossless two-terminal-pair network labeled with respect to ρ_1 , ρ_2 , and t . The symbols and parameters of Fig. 2 are defined by equations (3), (4), and (5).

In the next paragraphs attention is focused on those properties which a reflection coefficient must have to be realizable by a passive impedance.

The reflection coefficient ρ_1 can also be expressed as

$$\rho_1(s) = \frac{Z_1(s) - 1}{Z_1(s) + 1}, \quad (6)$$

where $Z_1(s)$ is the input impedance of the lossless two terminal-pair terminated in a one-ohm resistance. Since $Z_1(s)$ is the input impedance of a passive network, it is a positive-real (p.r.) function of the

complex frequency variable $s = \sigma + j\omega$. By a positive-real function of $s = \sigma + j\omega$ is meant a function which is real for s real and which has a non-negative real part for non-negative values of σ . It can be shown that the necessary and sufficient conditions on e_1 for $Z_1(s)$ to be a p.r. function are that $e_1(s)$ be a rational meromorphic function of s , regular in the right-half s -plane, and $|e_1(j\omega)| \leq 1$ (11).

Equation (3), the first part of equation (5), and Fig. 2 reveal that $1 - |e_1(j\omega)|^2$ is equal to $|t(j\omega)|^2$ and is the ratio of the power transmitted at real frequencies to the load (when $E_2 = 0$) to the power available at the generator. The above ratio is called the per-unit power transmitted. If $1 - |e_1(j\omega)|^2$ is the per-unit power transmitted to the load, $|e_1(j\omega)|^2$ can be considered the per-unit power rejected by the load. It is evident that neither $|e_1(j\omega)|$ nor $|t(j\omega)|$ can be greater than unity. Thus

$$|t(j\omega)|^2 = 1 - |e_1(j\omega)|^2 = 1 - |e_2(j\omega)|^2. \quad (7)$$

Since $e_1(s)$ is a rational meromorphic function of s , it is real for real s , and thus $e_1(j\omega)$ must be a real function of $j\omega$. The square of the magnitude of $e_1(j\omega)$ is an even real function of ω and, consequently, a real function of ω^2 . Equation (7) implies that similar statements can be made regarding $|e_2(j\omega)|^2$ and $|t(j\omega)|^2$. Thus, equation (7) can be restated as

$$t(j\omega)t(-j\omega) = 1 - e_1(j\omega)e_1(-j\omega) = 1 - e_2(j\omega)e_2(-j\omega). \quad (8)$$

By extending equation (8) into any part of the complex plane, it can be shown (12), (13)

$$t(s)t(-s) = -\rho_1(s)\rho_1(-s) = -\rho_2(s)\rho_2(-s). \quad (9)$$

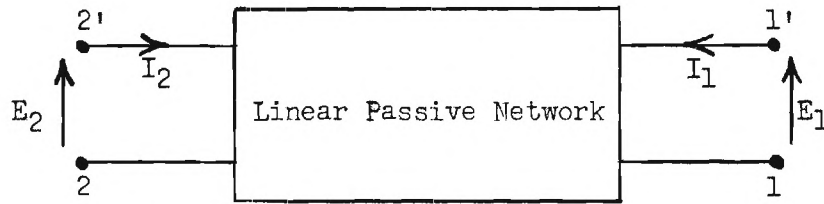
Behavior of ρ_1 and t in terms of open-circuit impedances.--A two-terminal-pair network can be described in several ways in terms of behavior at the terminal-pairs. The reflection and transmission coefficients of one-ohm terminated networks represent but one of the useful means of portraying the network characteristics. The so-called open-circuit impedances convey the same amount of information in a more convenient manner for certain purposes. It is desirable to describe the open-circuit impedances and to show relationships with the reflection and transmission coefficients.

The functions z_{11} , z_{22} , and z_{12} are the open-circuit impedances. The parameter z_{11} is the input impedance at the "1" terminal-pair with the "2" terminal-pair open-circuited. A similar type of description identifies z_{22} . The parameter z_{12} is the ratio of voltage at the "2" terminal-pair to the input current at the "1" terminal-pair, and conversely. The impedances z_{11} and z_{22} are driving-point impedances, while z_{12} is a transfer impedance. The open-circuit impedances are shown with appropriate network designations in Fig. 3. If a two terminal-pair is terminated by a one-ohm resistance, the input impedance is (14)

$$Z_1 = \frac{(z_{11}z_{22} - z_{12}^2) + z_{11}}{z_{22} + 1}. \quad (10)$$

It is also possible to specify Z_1 as

$$Z_1 = \frac{m_1 + n_1}{m_2 + n_2} = \frac{P(s)}{Q(s)}, \quad (11)$$



$$z_{22}=E_2/I_2 \text{ with } I_1=0, \quad z_{11}=E_1/I_1 \text{ with } I_2=0, \\ z_{12}=z_{21}=E_2/I_1 \text{ with } I_2=0, \text{ and } E_1/I_2 \text{ with } I_1=0.$$

Fig. 3. Open-circuit Impedances

where the m 's and n 's are functions of s and are the even and odd parts, respectively, of $P(s)$ and $Q(s)$. Comparison of equations (10) and (11) reveals that two sets of relationships exist between the m 's and n 's and the open-circuit impedances. Guillemin (15) outlines the method for finding both sets of relationships and distinguishes them as cases A and B. For case A

$$Z_{11} = \frac{m_1}{n_2}, \quad Z_{22} = \frac{m_2}{n_2}, \quad Z_{12} = \frac{\sqrt{m_1 m_2 - n_1 n_2}}{n_2}. \quad (12)$$

For case B

$$Z_{11} = \frac{n_1}{m_2}, \quad Z_{22} = \frac{n_2}{m_2}, \quad Z_{12} = \frac{\sqrt{n_1 n_2 - m_1 m_2}}{m_2}. \quad (13)$$

Case A by definition corresponds to $m_1 m_2 > n_1 n_2$ for real s values, while case B corresponds to $n_1 n_2 > m_1 m_2$ for real s values. It can be shown that the open-circuit impedances of either equation (12) or (13) are all physically realizable as reactance functions provided that the term under the radical of either equation (12) or (13) is a perfect square

in s . In the event that it is not, it can be shown (16) that the numerator and denominator of the original expression for $Z_1(s)$ can be multiplied by the term (m_0+n_0) from

$$(m_0+n_0)(m_0-n_0) = (m_0^2 - n_0^2), \quad (14)$$

where $(m_0^2 - n_0^2)$ is equal to the factors which occurred with odd multiplicity in the original term under the radical of either equation (12) or (13), depending on whether case A or case B is applicable. The term (m_0+n_0) contains no right-half s -plane zeros. The revised open-circuit impedances which result from the revised $Z_1(s)$ are physically realizable because z_{12} now has a term under the radical which is a perfect square in s .

It is possible to express ρ_1 in terms of m_1 , n_1 , m_2 , and n_2 . For case A

$$\rho_1 = \frac{Z_1 - 1}{Z_1 + 1} = \frac{(m_1 - m_2) + (n_1 - n_2)}{(m_1 + m_2) + (n_1 + n_2)}. \quad (15)$$

Similarly,

$$\rho_2 = \frac{Z_2 - 1}{Z_2 + 1} = \frac{-(m_1 - m_2) + (n_1 - n_2)}{(m_1 + m_2) + (n_1 + n_2)}. \quad (16)$$

In terms of the open-circuit impedances, t can be written as

$$t = \frac{2Z_{12}}{(Z_{11}Z_{22} - Z_{12}^2) + Z_{11} + Z_{22} + 1}. \quad (17)$$

The derivation of equation (17) is found through application of equation

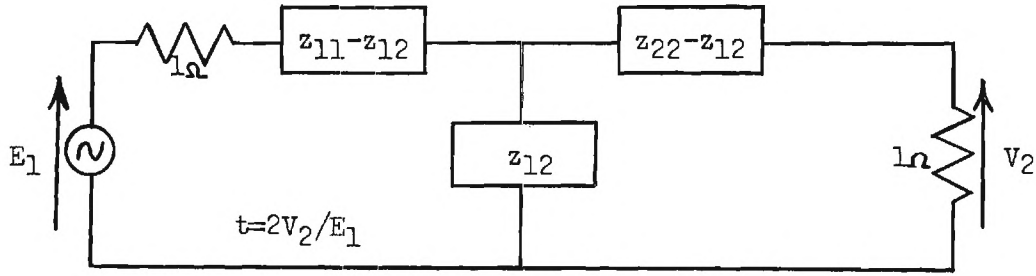


Fig. 4. Tee Representation of Two Terminal-pair

(5) to a tee representation of a one-ohm terminated network as shown in Fig. 4.

For case A, in particular, t becomes

$$t = \frac{2 \sqrt{m_1 m_2 - n_1 n_2}}{(m_1 + m_2) + (n_1 + n_2)} . \quad (18)$$

Several interesting facts about the behavior for case A are brought forth by equations (15), (16), and (18). The denominators of e_1 , e_2 , and t are alike, indicating that e_1 , e_2 , and t have the same poles, unless a cancellation of identical factors in numerators and denominators takes place. However, if no cancellation of factors takes place and e_1 and e_2 have the same denominators, the numerators of e_1 and e_2 have identical odd parts while the even parts are opposite in sign. Comparison of equations (12) and (18) shows that the numerator of t is identical with the numerator of z_{12} except for a constant multiplier.

For case B,

$$e_1 = \frac{Z_1 - 1}{Z_1 + 1} = \frac{(m_1 - m_2) + (n_1 - n_2)}{(m_1 + m_2) + (n_1 + n_2)} . \quad (19)$$

Equation (13) can be used to obtain ρ_2 for case B as

$$\rho_2 = \frac{Z_2 - 1}{Z_2 + 1} = \frac{(m_1 - m_2) - (n_1 - n_2)}{(m_1 + m_2) + (n_1 + n_2)}. \quad (20)$$

The expression for t for case B becomes

$$t = \frac{2\sqrt{n_1 n_2 - m_1 m_2}}{(m_1 + m_2) + (n_1 + n_2)}. \quad (21)$$

The same facts about ρ_1 , ρ_2 , and t which are noted for case A are evident for case B with the obvious exception that the numerators of ρ_1 and ρ_2 differ only by the sign of the odd parts. Comparisons of equation (12) with (18) and equation (13) with (21) show that the numerator of t is identical with the numerator of z_{12} except for a constant multiplier. Since it was indicated earlier that z_{12} is a reactance function, the numerator of z_{12} is either an even or an odd polynomial of s . When the numerator of z_{12} is an even polynomial in s , as it is for case A, the numerator of t is also an even polynomial in s . For case B the numerator of z_{12} is an odd polynomial in s , and the numerator of t is an odd polynomial in s .

It can be shown that the numerator of $t(s)t(-s)$ must be a perfect square in s and that the zeros of $t(s)$ occur in pairs symmetric with respect to the origin (17).

Zero and pole forms of ρ_1 and ρ_2 .—Alternate forms for ρ_1 and ρ_2 can be developed in terms of their zeros and poles in the s -plane. The general expression for ρ_1 in terms of zeros and poles is

$$\rho_1 = h \frac{(s-s_{o1})(s-s_{o2})\dots(s-s_{on})}{(s-s_{p1})(s-s_{p2})\dots(s-s_{pn})}. \quad (22)$$

Comparisons of equation (15) with (16) and equation (19) with (20) indicate that the general expressions for ρ_2 for cases A and B, in order, are

$$\rho_2 = h(-1)^{1+n} \frac{(s+s_{o1})(s+s_{o2})\dots(s+s_{on})}{(s-s_{p1})(s-s_{p2})\dots(s-s_{pn})}, \quad (23)$$

$$\rho_2 = h(-1)^n \frac{(s+s_{o1})(s+s_{o2})\dots(s+s_{on})}{(s-s_{p1})(s-s_{p2})\dots(s-s_{pn})}. \quad (24)$$

Aside from a difference in sign, ρ_2 for case A is of the same form as ρ_2 for case B. This is consistent with the previous discussion of changes in sign of the even or odd parts of ρ_2 numerators as compared with ρ_1 numerators. Moreover, equations (22) through (24) place in evidence the unlike signs of the zeros of ρ_1 and ρ_2 .

In order for ρ_1 , ρ_2 , and t to be the reflection and transmission coefficients of a passive network, the poles of ρ_1 , ρ_2 , and t must not lie in the right-half s -plane. The zeros of ρ_1 , ρ_2 , and t can occupy positions anywhere in the complex s -plane without violating passivity or stability conditions (18).

Equation (9) can be written as

$$\rho_1(s)\rho_1(-s) = 1 - t(s)t(-s), \quad (25)$$

$$\rho_2(s)\rho_2(-s) = 1 - t(s)t(-s), \quad (26)$$

which suggest that ρ_1 (or ρ_2) can be obtained from t , and vice versa.

Let it be assumed, for example, that $\varrho_1(s)$ is known and a $t(s)$ which meets the realizability conditions is desired. Equation (25) indicates that $t(s)t(-s)$ can be obtained from $\varrho_1(s)$. As was shown previously, $t(s)t(-s)$ is a ratio of polynomials in s^2 and can thus be written as $X(-s^2)/Y(-s^2)$, where X and Y are real for s real. It can be shown (19) that the X/Y form can be rearranged as

$$t(s)t(-s) = \frac{X(-s^2)}{Y(-s^2)} = \frac{x(s)x(-s)}{y(s)y(-s)}. \quad (27)$$

The expression for $t(s)$ is chosen as

$$t(s) = \frac{x(s)}{y(s)}, \quad (28)$$

where $y(s)$ has only left-half s -plane zeros of $Y(-s^2)$ because of realizability conditions. The zeros of $x(s)$ can lie anywhere in the complex plane, the only requirement being that $x(s)x(-s)$ is a perfect square in s .

It is possible that the first choice of $\varrho_1(s)$ leads to a function for $t(s)$ and a consequent $t(s)t(-s)$, the numerator of which is not a perfect square. In this case, the numerator and denominator of $\varrho_1(s)$ are multiplied by the term (m_0+n_0) from

$$(m_0+n_0)(m_0-n_0) = (m_0^2 - n_0^2), \quad (29)$$

where $(m_0^2 - n_0^2)$ is equal to the factors of the original $X(-s^2)$ which occurred with odd multiplicity. The term (m_0+n_0) contains no right-half s -plane zeros. The revised $X(-s^2)$ is now a perfect square in s , and a

realizable $t(s)$ can be found. The above procedure is very similar to the previously described method for obtaining a physically realizable z_{12} .

The conditions which a function of s must satisfy in order to be a reflection coefficient of a physically realizable, lossless two terminal-pair terminated in a one-ohm resistance can be summarized at this point. The function $\rho(s)$ must be a rational meromorphic function of s which is regular in the right-half s -plane and $|\rho(j\omega)| \leq 1$.

Right-half s -plane Zeros of Reflection Coefficients

If a reflection coefficient, for example, ρ_1 , has right-half s -plane zeros, the reversal of sign between the zeros of ρ_1 and ρ_2 , shown in equations (22) and (23), indicates that ρ_2 has left-half s -plane zeros diametrically opposite to the right-half s -plane zeros of ρ_1 . This is true when ρ_2 has the same denominator as ρ_1 . The two reflection coefficients always have the same denominator unless cancellation of common factors between numerators and denominators takes place.

In this study, there are several possible ways in which right-half s -plane zeros can appear in reflection coefficients of the complete system. First, a reflection coefficient might contain a right-half s -plane zero of reflection at the outset; second, simple real zeros or sets of simple conjugate zeros might be added intentionally; and, third, double multiplicity real zeros and sets of double multiplicity conjugate zeros might be added intentionally. While there can exist in any one problem the combination of any of the above cases, the second and third cases are of primary interest herein. If at real frequencies the magnitudes of a reflection coefficient and the logarithm

of the reciprocal of that reflection coefficient are to remain unchanged when right-half s-plane reflection zeros are added, it is necessary to add left-half s-plane poles of reflection at the diametrically opposite points in the s-plane. This can be shown by reference to equation (24). Thus, in effect, the addition of right-half s-plane zeros of reflection leaves the magnitude of a reflection coefficient unchanged at real frequencies and, at most, introduces a change in angle at real frequencies. The details of three cases are outlined below.

Right-half s-plane zeros of reflection in an existing network.--Suppose

ρ_1 has the form

$$\rho_1 = h \frac{(s-s_{o1})(s-s_{o2})\dots(s-s_{on})}{(s-s_{p1})(s-s_{p2})\dots(s-s_{pn})}, \quad (30)$$

where some of the s_{oi} 's lie in the right-half s-plane. By equation (23), the corresponding ρ_2 is

$$\rho_2 = \pm h \frac{(s+s_{o1})(s+s_{o2})\dots(s+s_{on})}{(s-s_{p1})(s-s_{p2})\dots(s-s_{pn})}, \quad (31)$$

where the right-half s-plane zeros of ρ_1 have counterparts in the corresponding left-half s-plane zeros of ρ_2 . It is not necessary for the transmission coefficient t to contain any of the same right-half s-plane zeros as ρ_1 or ρ_2 . For proof, it is needed to show only one network in which the statement is verified. Such a network is shown in Fig. 5, wherein it is seen that $Z_1(s)$ is given by

$$Z_1(s) = \frac{2s+1}{2s^2+s+1}. \quad (32)$$

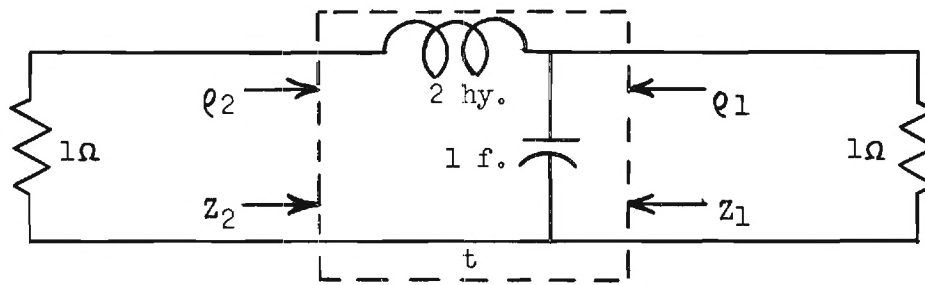


Fig. 5. Simple Network

Equations (15), (16), and (18) are used to find

$$e_1(s) = -2 \frac{(s-0.5)(s-0)}{(2s^2+3s+2)}, \quad (33)$$

$$e_2(s) = +2 \frac{(s+0.5)(s+0)}{(2s^2+3s+2)}, \quad (34)$$

$$t(s) = \frac{2}{(2s^2+3s+2)}. \quad (35)$$

Reference to equations (33), (34), and (35) verifies the fact that t does not have the same zeros as e_1 or e_2 .

Addition of simple right-half s-plane zeros of reflection.--Right-half s-plane zeros can lie either on the positive real axis or occur in complex conjugate pairs. If e_1 is in the form $[A(s)/B(s)]$ (where $A(s)$ and $B(s)$ are polynomials in s) prior to adding a right-half s-plane zero, $t(s)$ can be described in terms of

$$t(s)t(-s) = 1 - \rho_1(s)\rho_1(-s) = \quad (36)$$

$$\frac{B(s)B(-s) - A(s)A(-s)}{B(s)B(-s)} .$$

In accordance with a previous discussion, the numerator of the right-hand side of equation (36) must be a perfect square in s . (It is assumed that ρ_1 is adjusted, if necessary, to make this so.)

When a simple zero at $s = -\mathfrak{S}_1$ is added to a reflection coefficient $A(s)/B(s)$ in such a way that the magnitude remains unchanged at real frequencies, the result has the modified form

$$\rho_{1(mod)} = \frac{A(s)(s - \mathfrak{S}_1)}{B(s)(s + \mathfrak{S}_1)} . \quad (37)$$

According to equation (36), a corresponding expression for $t(s)t(-s)$ is

$$[t(s)t(-s)]_{(mod)} = \frac{[B(s)B(-s) - A(s)A(-s)](-s + \mathfrak{S}_1)(s + \mathfrak{S}_1)}{B(s)B(-s)(s + \mathfrak{S}_1)(-s + \mathfrak{S}_1)} . \quad (38)$$

If the denominator of $t(s)_{(mod)}$ is to have the same roots as the denominator of $\rho_{1(mod)}$, the common numerator and denominator factors in equation (38) cannot be canceled. Equation (38), however, cannot qualify as a realizable expression, since $[(-s + \mathfrak{S}_1)(s + \mathfrak{S}_1)]$ is not a perfect square in s , even though $[B(s)B(-s) - A(s)A(-s)]$ is. The difficulty can be resolved by respecifying $\rho_{1(mod)}$ as

$$\rho_{1(mod)} = \frac{A(s)(s - \mathfrak{S}_1)(s + \mathfrak{S}_1)}{B(s)(s + \mathfrak{S}_1)(s + \mathfrak{S}_1)} , \quad (39)$$

which is the $\rho_{1(mod)}$ of equation (37) multiplied by the unity factor $[(s+\mathfrak{I}_1)/(s+\mathfrak{I}_1)]$. The corresponding $t(s)t(-s)_{(mod)}$ becomes

$$[t(s)t(-s)]_{(mod)} = \frac{[B(s)B(-s)-A(s)A(-s)](-s+\mathfrak{I}_1)^2(s+\mathfrak{I}_1)^2}{B(s)B(-s)(s+\mathfrak{I}_1)^2(-s+\mathfrak{I}_1)^2}. \quad (40)$$

It is seen that the numerator of equation (40) is a perfect square in s .

If case A is considered, the expression for the original reflection coefficient ρ_2 can be shown to be $[-A(-s)/B(s)]$. The $\rho_{2(mod)}$, as obtained from equation (39) is

$$\rho_{2(mod)} = \frac{-A(-s)(-s-\mathfrak{I}_1)(-s+\mathfrak{I}_1)}{B(s)(s+\mathfrak{I}_1)(s+\mathfrak{I}_1)}. \quad (41)$$

Comparison of equation (41) with equation (39) reveals that the addition to ρ_1 of a simple real zero in the right-half s -plane results in the original ρ_1 and ρ_2 being multiplied by factors which introduce a left-half s -plane zero of reflection, a right-half s -plane zero of reflection, and two left-half s -plane poles of reflection.

A set of simple conjugate right-half s -plane reflection zeros can be added in much the same way. In this case, the revised reflection coefficients are

$$\rho_{1(mod)} = \frac{A(s)(s-s_\nu)(s-\bar{s}_\nu)(s+s_\nu)(s+\bar{s}_\nu)}{B(s)(s+s_\nu)(s+\bar{s}_\nu)(s+s_\nu)(s+\bar{s}_\nu)}, \quad (42)$$

$$\rho_{2(mod)} = \frac{-A(s)(-s-s_\nu)(-s-\bar{s}_\nu)(-s+s_\nu)(-s+\bar{s}_\nu)}{B(s)(s+s_\nu)(s+\bar{s}_\nu)(s+s_\nu)(s+\bar{s}_\nu)}. \quad (43)$$

Again, it should be noted that the same added zeros and poles are in-

cluded in $\rho_{2(\text{mod})}$ as are included in $\rho_{1(\text{mod})}$.

Equation (40) shows that the process of adding a simple zero to a reflection coefficient at $s=\mathfrak{S}_1$ in such a way that the magnitude is unchanged at real frequencies results in the addition of zeros of transmission at $s=\pm\mathfrak{S}_1$ to t . Since, as explained previously, the zeros of a realizable transmission coefficient must be symmetrically arranged with respect to the origin, the revised transmission coefficient must contain one additional zero at $+\mathfrak{S}_1$. This fact is of practical significance later when the synthesis of networks with added right-half s-plane reflection zeros is considered.

Addition of double multiplicity right-half plane zeros of reflection.--

The addition of double right-half s-plane zeros of reflection can be made in exactly the same manner as the addition of the simple zeros. However, it is shown below in equation (45) that it is not necessary to introduce an additional factor to insure that the numerator of $[t(s)t(-s)]_{(\text{mod})}$ is a perfect square. The expression for $\rho_{1(\text{mod})}$ with an added double zero on the positive real axis is

$$\rho_{1(\text{mod})} = \frac{A(s)(s-\sigma_1)^2}{B(s)(s+\sigma_1)^2}. \quad (44)$$

The corresponding $[t(s)t(-s)]_{(\text{mod})}$ is

$$[t(s)t(-s)]_{(\text{mod})} = \frac{[B(s)B(-s)-A(s)A(-s)](s+\sigma_1)^2(s+\sigma_1)^2}{B(s)B(-s)(s+\sigma_1)^2(s+\sigma_1)^2}, \quad (45)$$

while $\rho_{2(\text{mod})}$ becomes

$$\rho_{2(mod)} = \frac{-A(s)(s+\sigma_1)^2}{B(s)(s+\sigma_1)^2} \quad (46)$$

The concept of a double zero of reflection is used to distinct advantage in the matching problem considered later for specifying the limits of tolerance and gain.

Origin of the double reflection zero.--If the common factors in the numerator and denominator of equation (46) are canceled and a realizable $t(s)$ is extracted from equation (45), there results

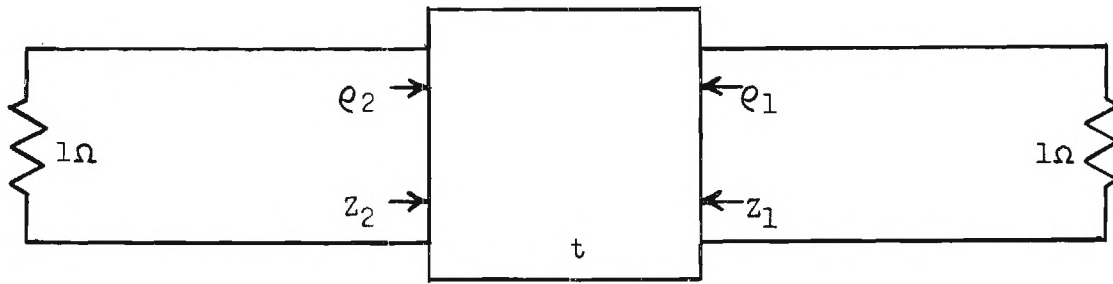
$$\rho_2 = \rho_{2(mod)} = \frac{-A(s)}{B(s)} \quad (47)$$

$$t(s)_{(mod)} = \frac{\sqrt{B(s)B(-s)-A(s)A(-s)} (s+\sigma_1)(-s+\sigma_1)}{B(s)(s+\sigma_1)(s+\sigma_1)} \quad (48)$$

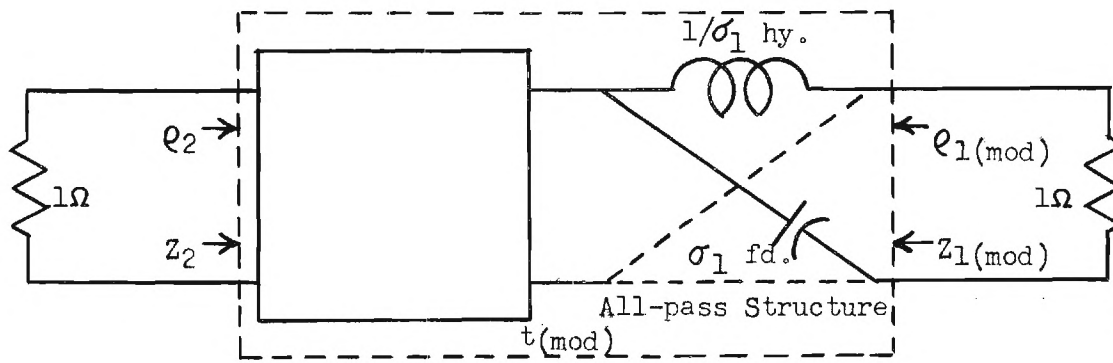
while $\rho_{1(mod)}$ remains as

$$\rho_{1(mod)} = \frac{A(s)(s-\sigma_1)^2}{B(s)(s+\sigma_1)^2} \quad (49)$$

A legitimate question at this point is what kind of modification can be made on a system to bring about the effects noted in equations (47) through (49), i.e., leave ρ_2 unchanged and at real frequencies cause a change in phase but no change in magnitude in ρ_1 and t . It can be shown that an all-pass structure in the form of a constant-resistance lattice with inductive and capacitive arms can be connected between the one-ohm load resistance and the original lossless network to produce the effects noted above. Fig. 6 shows the networks prior to and after



(a) Original System



(b) Modified System

Fig. 6. System with All-pass Structure

insertion of the all-pass structure.

Since the all-pass structure is of the constant-resistance type having a characteristic resistance of one ohm and is terminated on its right-hand side by a one-ohm resistance, the impedance Z_2 and the reflection coefficient e_2 are unchanged with the insertion of the lattice structure. At real frequencies, a well-known property of an all-pass constant-resistance lattice terminated in its characteristic resistance is that the input voltage and output voltage are equal in magnitude, but may differ in phase. If this property is applied to Fig. 6(b) and use is made of the last section of equation (5), it can be shown that

the introduction of the all-pass structure leaves the magnitude of the modified transmission coefficient the same as that of the original transmission coefficient at real frequencies and, at most, introduces a phase shift. However, the impedance and the reflection coefficient looking to the left from the right-hand resistance are altered by the presence of the all-pass structure. By using the parameter values of Fig. 6, the value of $Z_{1(mod)}$ becomes

$$Z_{1(mod)} = \frac{2 + Z_1 \left(\frac{\sigma_1}{s} + \frac{s}{\sigma_1} \right)}{2Z_1 \left(\frac{\sigma_1}{s} + \frac{s}{\sigma_1} \right)} . \quad (50)$$

If the original Z_1 is in the form

$$Z_1 = \frac{(m_1 + n_1)}{(m_2 + n_2)} , \quad (51)$$

the original ρ_1 is in the form

$$\rho_1 = \frac{(m_1 - m_2) + (n_1 - n_2)}{(m_1 + m_2) + (n_1 + n_2)} . \quad (52)$$

The new impedance $Z_{1(mod)}$ becomes

$$Z_{1(mod)} = \frac{\left[\left(s^2 + \frac{1}{\sigma_1^2} \right) m_1 + \frac{2s}{\sigma_1} n_2 \right] \left[\frac{2s}{\sigma_1} m_2 + \left(s^2 + \frac{1}{\sigma_1^2} \right) n_1 \right]}{\left[\left(s^2 + \frac{1}{\sigma_1^2} \right) m_2 + \frac{2s}{\sigma_1} n_1 \right] \left[\frac{2s}{\sigma_1} m_1 + \left(s^2 + \frac{1}{\sigma_1^2} \right) n_2 \right]} . \quad (53)$$

The $\rho_{1(mod)}$ becomes

$$\rho_{1(mod)} = \frac{[(m_1 - m_2) - (n_1 - n_2)] (s - \sigma_1)^2}{[(m_1 + m_2) + (n_1 + n_2)] (s + \sigma_1)^2} . \quad (54)$$

Equation (54) indicates that the insertion of an all-pass constant-resistance structure produces the desired double zero of reflection.

In practice, the resistance termination usually cannot be separated from the first elements of the lossless network. Hence, the direct insertion of the all-pass lattice generally cannot be made. It is possible, however, to electrically "imbed" an all-pass structure inside the matching network in such a way that the effects are identical with those noted above for the lattice and resistance combination. This procedure is described in Chapter III.

CHAPTER III

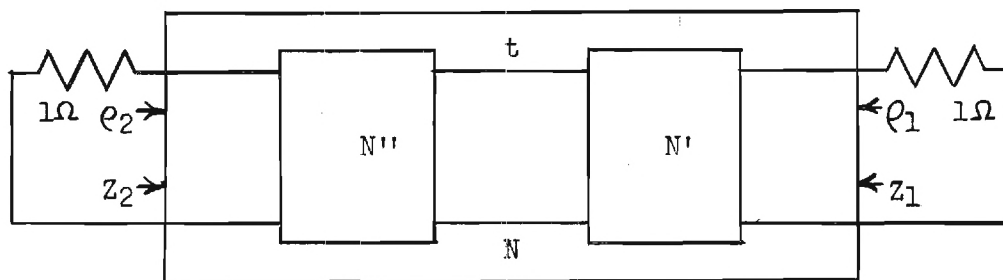
PROCEDURE

Conditions of Physical Realizability

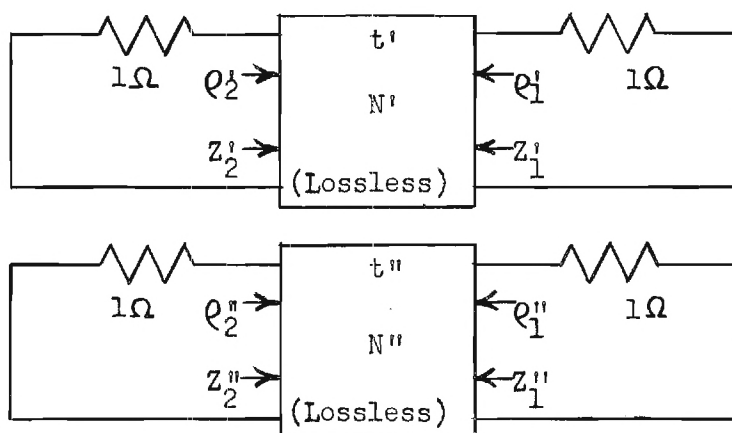
As a prelude to examining the conditions of physical realizability of the reflection coefficients for complete systems of matching networks and arbitrary source and load impedances, it is instructive to consider conditions governing the reflection and transmission coefficients for systems consisting of matching networks and arbitrary load impedances. Fig. 7 shows the arrangement of a duo-cascade of two lossless networks terminated in one-ohm resistances. The two lossless networks can be grouped as one network with an overall transmission coefficient t and reflection coefficients ρ_1 and ρ_2 . The load and matching networks are identified by N' and N'' , respectively, in Fig. 7. The networks N' and N'' are lossless two terminal-pairs and, as such, can be described in terms of their individual transmission and reflection coefficients on a one-ohm terminated basis. Fig. 7 shows this arrangement also. For identification purposes, the right-hand end of each network bears a "1" designation, while the left-hand end bears a "2" designation.

From equations (A-15) through (A-17) in Appendix A, the appropriate equations for describing the overall behavior in terms of the transmission and reflection coefficients of N' and N'' are

$$\rho_1 = \rho_1' + \rho_1'' \frac{(t')^2}{1 - \rho_2' \rho_1''} , \quad (55)$$



(a) Duo-cascade



(b) Individual Networks

Fig. 7. Duo-cascade of Two Terminal-pairs and Individual Networks

$$e_2 = e_2'' + e_2' \frac{(t')^2}{1 - e_2' e_1''}, \quad (56)$$

$$t = \frac{t' t''}{1 - e_2' e_1''}. \quad (57)$$

The following theorems point out pertinent facts concerning the network behavior as described by equations (55) through (57).

Theorem 1: If t' has a zero in the right-half s -plane or on the

imaginary axis, t must have the same zero with at least the same multiplicity.

Proof: The denominator of equation (57) is $(1 - e_2' e_1'')$. With the aid of equations (15) and (16), the denominator can be expressed as

$$(1 - e_2' e_1'') = \frac{2Z_2' + 2Z_1''}{(Z_2' + 1)(Z_1'' + 1)} = \frac{1}{\frac{1}{2} \left(\frac{Z_2' Z_1''}{Z_2' + Z_1''} \right) + \frac{1}{2} + \frac{1}{2} \left(\frac{1}{Z_2' + Z_1''} \right)} \quad (58)$$

The denominator of the last expression of equation (58) is the sum of three expressions each of which is a p.r. function of s . This denominator is thereby positive real. The expression $(1 - e_2' e_1'')$ is the reciprocal of a p.r. function and is also positive real (20). It follows that $(1 - e_2' e_1'')$ has neither poles nor zeros in the right-half s -plane. Thus, $1/(1 - e_2' e_1'')$ cannot vanish for any right-half s -plane values, leaving the t of equation (57) with the right-half s -plane zeros of t' .

If $(1 - e_2' e_1'')$ has a zero on the imaginary axis at $j\omega_0$, the zero must be simple because of the positive real nature of $(1 - e_2' e_1'')$. The absolute values $|e_2'(j\omega)|$ and $|e_1''(j\omega)| = 1$ at $s = j\omega_0$. As shown in equation (7), when $|e_2'(j\omega)| = 1$, $t' = 0$, and when $|e_1''(j\omega)| = 1$, $t'' = 0$. The order of the zeros of the product $t' t''$ at $s = j\omega_0$ is reduced by one by the simple zero of $(1 - e_2' e_1'')$. For sake of argument, the zero can be assumed to be that of t'' , leaving t with at least the j -axis zeros of t' .

The vanishing of the denominator of equation (57) at a zero of

transmission leads to what is called a degenerate zero and is of sufficient importance to warrant a detailed discussion later in this chapter. Zeros of $(1-\rho_2'\rho_1'')$ can be of arbitrary multiplicity in the left-half s-plane and need not be present in t' (21); however, the previously mentioned requirement that the zeros of t' be arranged symmetrically about the origin demands that a matching right-half s-plane zero appear for each left-half s-plane zero of t' .

Theorem 2: At any point s_0 in the right-half s-plane or on the imaginary axis at which t' has a zero of multiplicity n' , the reflection coefficient ρ_1 and its first $(2n'-1)$ derivatives are equal to ρ_1' and its first $(2n'-1)$ derivatives and, therefore, are independent of the network N'' , except for the degenerate case mentioned above.

Proof: Equation (55) can be rewritten as

$$e_1 = e_1' + (s-s_0)^{2n'} A(s), \quad (59)$$

where $A(s)$ is a rational fraction which does not contain poles at s equal to s_0 . Successive derivatives yield

$$\frac{de_1}{ds} = \frac{de_1'}{ds} + 2n'(s-s_0)^{2n'-1} A(s) + (s-s_0)^{2n'} \frac{dA}{ds}, \quad (60)$$

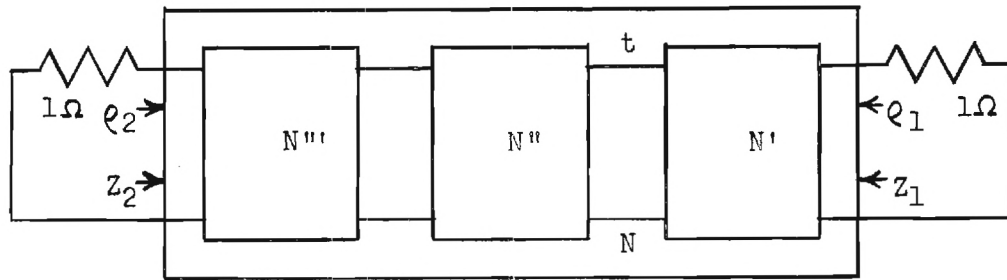
$$\frac{d^{2n'-1}e_1}{ds^{2n'-1}} = \frac{d^{2n'-1}e_1'}{ds^{2n'-1}} + (2n')(2n'-1) \frac{d^{2n'-2}A}{ds^{2n'-2}} + \frac{d^{2n'-1}A}{ds^{2n'-1}} \quad (61)$$

$$+ (2)(s-s_0)A(s) + (s-s_0)^2 B(s),$$

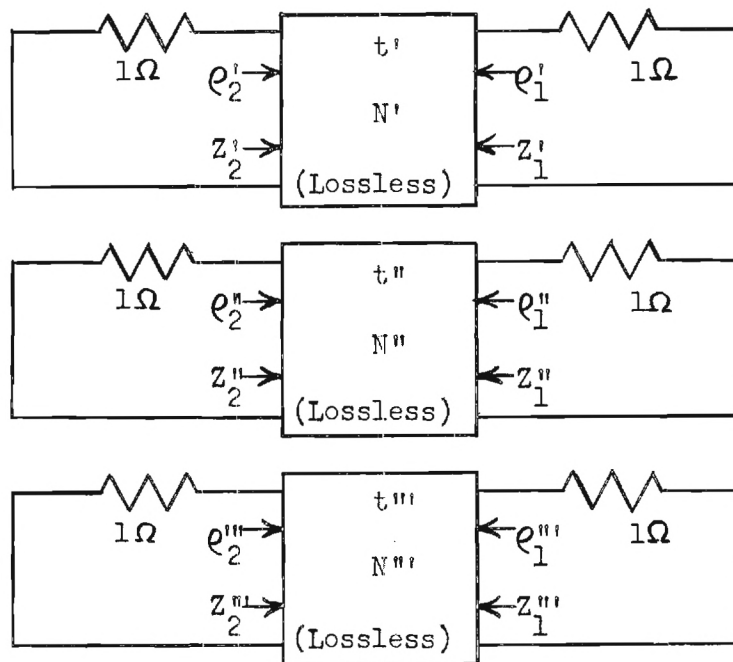
$$\frac{d^{2n'} \rho_1}{ds^{2n'}} = \frac{d^{2n'} \rho_1'}{ds^{2n'}} + (2n')(2n'-1) \frac{d^{2n'-2} \rho_1'}{ds^{2n'-2}} + \dots + (2)A(s) + (s-s_0)C(s), \quad (62)$$

where $A(s)$, $B(s)$, and $C(s)$ are rational fractions. At $s=s_0$, the derivatives of ρ_1 equal the corresponding derivatives of ρ_1' , up to and including the $(2n'-1)$ th derivative. This is evident from the vanishing of the $(s-s_0)$ factors at $s=s_0$ in the $2n'$ equations represented by equations (60) and (61) and dashed lines between equations (60) and (61). If $A(s)$ does not have a zero at $s=s_0$, then equation (62) has a non-vanishing term in addition to the $2n'$ th derivative, indicating that the $2n'$ th derivative and all succeeding derivatives may not be independent of the network N'' .

The above discussion can be extended to apply specifically to combinations of the load, matching, and source networks of this study, such as shown in Fig. 1. The three lossless networks can be grouped as a single network with an overall transmission coefficient t and reflection coefficients ρ_1 and ρ_2 . The terminations are one-ohm resistances in accordance with the previously adopted convention. The load, matching, and source networks are lossless two terminal-pairs and, as such, can be described in terms of their individual transmission and reflection coefficients on a one-ohm terminated basis. Fig. 8 shows the combined arrangement and the individual networks. The designation N represents the cascade of the three networks, N' represents the load network, N'' represents the matching network, and N''' represents the source network. As in the case of the duo-cascade, the



(a) Trio-cascade



(b) Individual Networks

Fig. 8. Trio-cascade of Two Terminal-pairs and Individual Networks

right-hand end of each network bears a "1" designation, while the left-hand end bears a "2" designation.

The equations for t , e_1 , and e_2 are developed in Appendix A in terms of the reflection and transmission coefficients of N' , N'' , and N''' and are listed below as

$$e_1 = e_1' + [e_1''(1 - e_2''e_1''') + (t'')^2 e_1'''] \cdot \quad (63)$$

$$\frac{(t')^2}{(1 - e_2'e_1'')(1 - e_2''e_1''') - e_2'e_1'''(t'')^2} ,$$

$$e_2 = e_2''' + [e_2''(1 - e_1''e_2') + (t'')^2 e_2'] \cdot \quad (64)$$

$$\frac{(t')^2}{(1 - e_2'e_1'')(1 - e_2''e_1''') - e_2'e_1'''(t'')^2} ,$$

$$t = \frac{t't''t'''}{(1 - e_2'e_1'')(1 - e_2''e_1''') - e_2'e_1'''(t'')^2} \cdot \quad (65)$$

The equations for the trio-cascade, equations (63) through (65), can be examined in the same manner as the duo-cascade equations. However, it is necessary to consider both reflection coefficients e_1 and e_2 , inasmuch as the two "outer" networks of N are investigated. A series of applicable theorems and proofs follows.

Theorem 3: If t' has a zero in the right-half s -plane or on the imaginary axis, t must have the same zero with at least the same multiplicity.

Proof: When N' is the network under examination, the cascaded combinations of networks N''' and N'' can be considered as a single lossless network. Application of Theorem 1 completes the proof of Theorem 3.

Theorem 4: If t''' has a zero in the right-half s -plane or on the imaginary axis, t must have the same zero with at least the same multiplicity.

Proof: When N''' is the network under examination, the cascaded

combination of networks N'' and N' can be considered as a single lossless network. Application of Theorem 1 completes the proof of Theorem 4.

Theorem 5: If t' and t''' have coincident zeros in the right-half s -plane or on the imaginary axis, t must have the same zeros with at least the sum of the multiplicities of the coincident zeros except as noted below. The exception is the special case where N'' has only one zero of transmission, a simple zero on the j -axis, which is not only coincident with the transmission zeros of t' and t''' , but which forms degenerate zeros simultaneously with N' and N''' . The exception can occur only for transmission zeros on the j -axis, since degenerate zeros occur only for j -axis values of s .

Proof: The denominator of equation (65), i.e., $[(1 - e_2' e_1'') \cdot (1 - e_2'' e_1''') - e_1' e_1''' (t'')^2]$, has no zeros in the right-half s -plane. This can be shown by arranging the denominator of equation (65) in the form

$$\left[(1 - e_2'' e_1''') - e_1' \frac{e_2' (t'')^2}{1 - e_2' e_1''} \right] (1 - e_2' e_1'') \cdot \quad (66)$$

The terms $(1 - e_2' e_1'')$ and $(1 - e_2'' e_1''')$ in expression (66) cannot be zero in the right-half s -plane since (as was shown in the proof of Theorem 1) $(1 - e_2' e_1'')$ and $(1 - e_2'' e_1''')$ are p.r. functions. The only possible way in which expression (66) can be zero in the right-half s -plane is

$$(1 - e_2'' e_1''') = e_1' \frac{e_2' (t'')^2}{1 - e_2' e_1''} \cdot \quad (67)$$

Let it be assumed that equation (67) is true for at least one right-half

s-plane value. The complete network system can be represented as network N''' in cascade with network $N^{(A)}$, where network $N^{(A)}$ is the cascaded combination of networks N'' and N' . This arrangement is illustrated by Fig. 9. Application of equation (56) to network $N^{(A)}$ leads to

$$e_2^{(A)} - e_2'' = \frac{(t'')^2}{1 - e_2' e_1''} \quad (68)$$

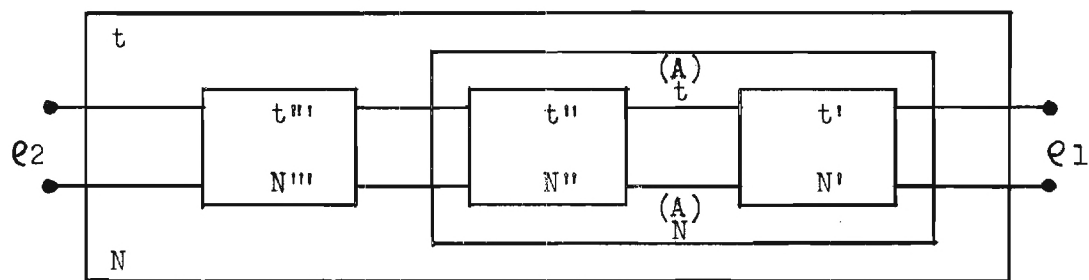
Combination of equations (67) and (68) indicates that

$$1 - \cancel{e_2'' e_1''} = e_1''' e_2^{(A)} - \cancel{e_1''' e_2''} \quad (69)$$

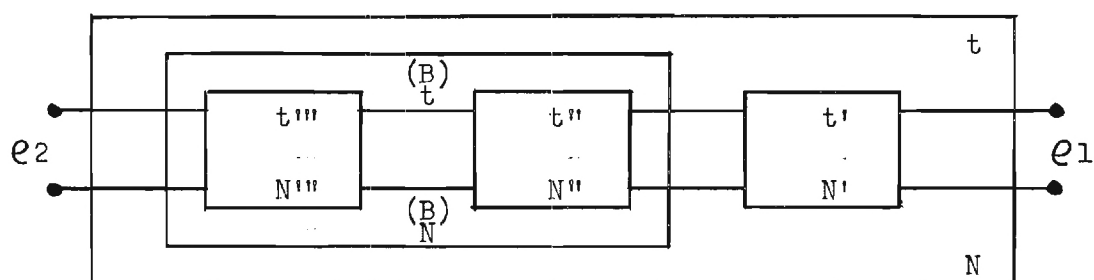
However, equation (69) cannot exist for any right-half s-plane value since $(1 - e_1''' e_2^{(A)})$ is itself positive real. Equation (67) leads to a contradiction. Thus, the denominator of equation (65) has no zeros in the right-half s-plane. Use of Theorems 3 and 4 leads to the conclusion that t contains the coincident right-half s-plane zeros of t' and t'' with a multiplicity at least equal to the sum of the multiplicities of the right-half s-plane zeros of t' and t'' .

When the denominator of the t of equation (65) has no j-axis zeros but t' and t'' have coincident j-axis zeros, it is evident from Theorems 3 and 4 that t contains the coincident zeros of t' and t'' with a multiplicity equal to at least the sum of their multiplicities.

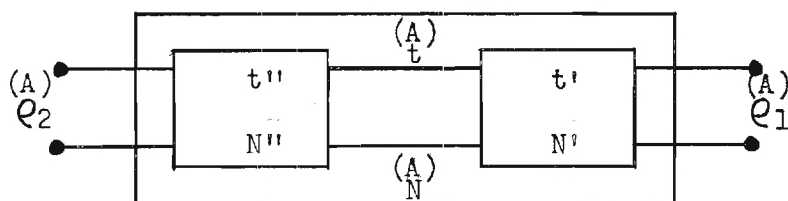
If the denominator of equation (65) is zero for a j-axis value of s , say $j\omega_0$, where t' and t'' have coincident zeros, the degenerate case results. A degenerate zero (or zeros) can occur when either $(1 - e_2' e_1'')$ or $(1 - e_2'' e_1''')$, or both, are zero at $j\omega_0$. It was shown in the



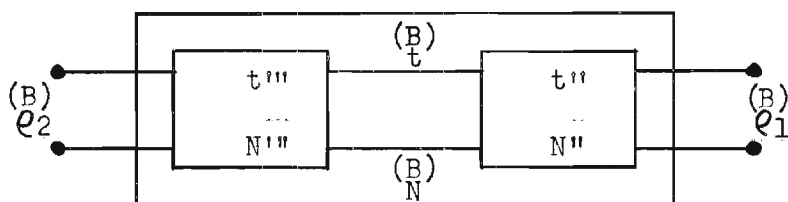
(a) Subnetwork $N^{(A)}$ within Cascade



(b) Subnetwork $N^{(B)}$ within Cascade



(c) Subnetwork $N^{(A)}$



(d) Subnetwork $N^{(B)}$

Fig. 9. Subnetworks and Locations within Cascade

proof of Theorem 1 that a j -axis zero of the function $(1-\rho_2'\rho_1'')$, for example, must be simple and can occur only when t'' also has a simple j -axis zero. A simple zero of either $(1-\rho_2'\rho_1'')$ or $(1-\rho_2''\rho_1''')$, but not both, at $j\omega_0$ cancels a simple zero of t'' , leaving t with the coincident zeros of t' and t''' with at least the sum of their multiplicities.

If N'' has at least two transmission zero-producing elements, and thereby at least a double zero of t'' , the vanishing of both $(1-\rho_2'\rho_1'')$ and $(1-\rho_2''\rho_1''')$ simply cancels a double zero of t'' , leaving t with the coincident zeros of t' and t''' with at least the sum of their multiplicities. In the extremely unlikely case (inserted here only for completeness) of a network N'' with a single transmission zero-producing element causing a simple zero of transmission at $j\omega_0$ where both $(1-\rho_2'\rho_1'')$ and $(1-\rho_2''\rho_1''')$ are zero, a zero is canceled from either t' or t''' , leaving t with the coincident zeros of t' and t''' with a multiplicity one less than the sum of the multiplicities of t' and t''' .

Theorem 6: At any point in the right-half s -plane or on the imaginary axis at which t' has a zero of multiplicity n' , the reflection coefficient ρ_1 and its first $(2n'-1)$ derivatives are equal to ρ_1' and its first $(2n'-1)$ derivatives, respectively, and, therefore, are independent of networks N'' and N''' , except for the degenerate case.

Proof: If the degenerate case is excluded and networks N'' and N''' are combined temporarily, the proof follows by application of Theorem 2.

Theorem 7: At any point in the right-half s -plane or on the imaginary axis at which t''' has a zero of multiplicity n''' , the reflection coefficient ρ_2 and its first $(2n'''-1)$ derivatives are equal to ρ_2'''

and its first $(2n'''-1)$ derivatives, respectively, and, therefore, are independent of networks N' and N'' , except for the degenerate case.

Proof: If the degenerate case is excluded and networks N'' and N' are combined temporarily, the proof follows by application of Theorem 2.

Necessary conditions from Taylor series expansions.--Briefly, theorems 5, 6, and 7 above state that t must have the same right-half s -plane or j -axis zeros of t' and t''' with at least the sum of the multiplicities of the zeros of t' and t''' and that at zeros of t' equalities exist term for term between ρ_1 and ρ_1' and their first $(2n'-1)$ derivatives and that at zeros of t''' equalities exist term for term between ρ_2 and ρ_2''' and their first $(2n'''-1)$ derivatives, except for degenerate cases. The conditions on t and the relationships between the constant terms and derivatives of the reflection coefficients are thus necessary conditions for the physical realizability of the trio-cascade network.

Since the return loss, i.e., the natural logarithm of the reciprocal of a reflection coefficient, is simply related to the reflection coefficient, the necessary conditions which stem from the equalities between the reflection coefficients and derivatives can be represented just as conveniently as equalities between corresponding return losses and derivatives. In this regard, it is pointed out that neither ρ_1 nor ρ_2 can be zero at a j -axis zero of t . This includes any transmission zeros at $s=0$ or $s=\infty$. Thus, $\ln(1/\rho_1)$, for example, is analytic at a transmission zero at $s=0$, $s=\infty$, and $s=j\omega_0$. There are circumstances, however, wherein a reflection coefficient and a transmission coefficient can be zero simultaneously at a right-half s -plane location. This be-

havior is described in Appendix B. In the latter case, the singularity of the return loss is removed, and the modified return loss is expanded in a Taylor series about the transmission zero. It is apparent that the constant term and the next $(2n'-1)$ coefficients of the Taylor series for $\ln(1/\rho_1)$ and $\ln(1/\rho_1')$ about a zero of t' are equal coefficient by coefficient and that the constant term and the next $(2n''-1)$ coefficients of the Taylor series for $\ln(1/\rho_2)$ and $\ln(1/\rho_2'')$ about a zero of t'' are equal coefficient by coefficient. The coefficients of the Taylor series for $\ln(1/\rho_1')$ and $\ln(1/\rho_2'')$ about transmission zeros are known since they are found directly from the known load (N') and source (N'') networks. The behavior of the overall reflection coefficients, ρ_1 and ρ_2 , is forced to conform to the necessary conditions, and any matching network derived from the overall reflection coefficients also must conform to the necessary conditions.

The locations for zeros of transmission considered in this study are:

- (a) at the origin ($s=0$),
- (b) at infinity ($s=\infty$),
- (c) on the imaginary axis ($s=\pm j\omega_v$),
- (d) on the real axis ($s=\sigma_v$),
- (e) at an arbitrary location in the right-half s -plane other than on the real axis ($s=\sigma_v \pm j\omega_v$).

Inasmuch as the basic methods for utilizing the necessary conditions to derive realizable reflection coefficients are essentially the same for all transmission zero locations, transmission zeros at infinity only are considered at this point and henceforward in the main body of this

study. Expressions for the coefficients of the Taylor series expansions of $\ln(1/q)$ about the other transmission zero locations are given in Appendix B. In addition, practical algebraic methods for calculating the coefficients are discussed in Appendix B.

The Taylor series expansions for $\ln(1/q_1)$ and $\ln(1/q_2)$ about a transmission zero at infinity can be rewritten from equation (B-9) as

$$\ln \frac{1}{q_1} = j\beta + \overset{(1)}{A}_1 \frac{1}{s} + \overset{(1)}{A}_3 \frac{1}{s^3} + \dots + \quad (70)$$

$$\overset{(1)}{A}_{2n'-3} \frac{1}{s^{2n'-3}} + \overset{(1)}{A}_{2n'-1} \frac{1}{s^{2n'-1}} + \dots,$$

$$\ln \frac{1}{q_2} = j\beta + \overset{(2)}{A}_1 \frac{1}{s} + \overset{(2)}{A}_3 \frac{1}{s^3} + \dots + \quad (71)$$

$$\overset{(2)}{A}_{2n''-3} \frac{1}{s^{2n''-3}} + \overset{(2)}{A}_{2n''-1} \frac{1}{s^{2n''-1}} + \dots,$$

where the superscript "(1)" or "(2)" identifies a coefficient with q_1 or q_2 . The $\overset{\infty}{A}$'s and β 's given in equations (70) and (71) are found from use of the necessary conditions for the realization of $\ln(1/q_1)$ and $\ln(1/q_2)$ when t' and t'' have zeros at infinity of multiplicity n' and n'' , respectively. The β 's are 0 or π , depending on the sign of the reflection coefficients at infinity. As shown in equations (B-10), (22), and (23), the $\overset{\infty}{A}$'s of equations (70) and (71) are related to the zeros and poles of q_1 by

$$\overset{(1)}{A}_{2m-1}^{\infty} = \frac{1}{2m-1} \left(\sum_1^{n'} (s_{oi})^{2m-1} - \sum_1^{n'} (s_{pi})^{2m-1} \right), \quad (72)$$

where $m=1, 2, 3, \dots, n'$, and

$$A_{2m-1}^{(2)\infty} = \frac{1}{2m-1} \left(-\sum_1^{n'''} (s_{oi})^{2m-1} - \sum_1^{n'''} (s_{pi})^{2m-1} \right), \quad (73)$$

where $m=1, 2, 3, \dots, n'''$. It is to be noted that while the A^∞ 's of equations (70) and (71) are known from N' and N''' , the s_{oi} 's and s_{pi} 's (the zeros and poles, respectively, of ρ_1) are not known but must be determined.

Degenerate zeros.--It is pointed out in the proof of Theorem 1 that a special case exists when degenerate zeros occur. The vanishing of the denominator in equation (63) at a j -axis zero of t' , for example, affects the $(2n'-1)$ th coefficient of the Taylor series for $\ln(1/\rho_1)$ in a very distinct way. It is mentioned in Appendix C that the value of the $(2n'-1)$ th coefficient is real and is always decreased by the presence of a degenerate zero of t' when n' is odd and increased by the presence of a degenerate zero when n' is even. The Taylor series for $\ln(1/\rho_2)$ is affected in the same way in that the value of the $(2n'''-1)$ th coefficient is always decreased by the presence of a degenerate zero of t''' when n''' is odd and increased when n''' is even.

It is shown in Appendix C that degenerate zeros can be caused only by the zero-producing elements of N'' nearest N' or N''' . Since the network N'' is to be derived, the choice of elements of N'' , including those which might produce degenerate zeros, is at the discretion of the network designer. Thus, a degenerate zero is not an entity which is an inevitable consequence of the statement of the problem, but something which can be introduced as desired when such introduction seems

advantageous.

Sufficiency conditions.--If it can be demonstrated that the necessary conditions imposed on ρ_1 , ρ_2 , and t by ρ_1' , ρ_2'' , t' , and t'' are all that are needed to obtain a physically realizable function for ρ_1 and ρ_2 , the necessary conditions are also sufficient.

Fano (22), using methods and proofs developed by Darlington (23), shows that for a given ρ_1' , ρ_2' , and t' , for example, a network can be constructed which has ρ_1' and ρ_2' for reflection coefficients and t' for a transmission coefficient.

To show sufficiency, the complete network system is divided into two parts, one part containing the sections representing the zeros of transmission of N' and the other part containing sections representing the remainder of the transmission zeros of N . Included necessarily in the second part are sections representing the transmission zeros of N'' . In Fig. 9, the above parts are identified with networks N' and $N^{(B)}$, respectively. At zeros of transmission, the equalities between $\ln(1/\rho_1')$ and $\ln(1/\rho_1)$ and the term by term equalities between certain coefficients from their Taylor series expansions indicate that network N' can be extracted from the overall system, leaving a known reflection coefficient $\rho_1^{(B)}$ for N .

However, except for sign, $\rho_2^{(B)}$ can be found immediately from $\rho_1^{(B)}$. Network $N^{(B)}$ can be divided now into two parts, one part, the left part, containing all the sections representing the transmission zeros of N'' and the remaining part, network N'' (the right part), containing the sections representing the transmission zeros of $N^{(B)}$ which are contained in neither N' nor N'' . At zeros of transmission, the resulting

equalities between $\ln(1/\rho_2^{(B)})$ and $\ln(1/\rho_2''')$ and the term by term equalities between certain coefficients from their Taylor series expansions about transmission zeros indicate that network N''' can be extracted from N , leaving a known reflection coefficient ρ_2'' for network N'' . Network N'' can be found by direct application of Darlington's methods.

Gain Integrals

By means of contour integration, it is possible to establish a relationship between the behavior of the magnitude of ρ_1 and ρ_2 at real frequencies and the necessary conditions. The contour integrals are developed in Appendix D for various transmission zero locations. Particular emphasis in this section is placed on finding the relationships between the coefficients of the Taylor series expansions of $\ln(1/\rho_1)$ and $\ln(1/\rho_2)$ about infinity and the behavior of $\ln(1/|\rho_1|)$ and $\ln(1/|\rho_2|)$ at real frequencies. These integrals involving the behavior of $\ln(1/|\rho_1|)$ and $\ln(1/|\rho_2|)$ at real frequencies are often called gain integrals.

Development of integral forms for both ends of the network.--The reflection coefficients ρ_1 and ρ_2 can have zeros anywhere in the s -plane. As explained in Chapter II, the right-half s -plane zeros appear in forms such as simple zeros and double zeros. A general expression for ρ_1 is

$$\rho_1 = \pm h \frac{\prod_{i=1}^{n-k} (s - s_{oi}) \prod_{j=1}^k (s - s_{rj}) (s - \sigma)^2}{\prod_{i=1}^n (s - s_{pi}) (s + \sigma)^2}, \quad (74)$$

where the s_{oi} 's are left-half s -plane zeros of ρ_1 , the s_{pi} 's are left-half s -plane poles of ρ_1 , the s_{rj} 's are right-half s -plane zeros of ρ_1 , and σ is a positive, real number which establishes the location of the

added double multiplicity zero term. It should be noted that the factor $(s+\sigma)^2$ in the denominator of equation (74) has the same magnitude as $(s-\sigma)^2$ at real frequencies, insuring that the added double zero does not affect the absolute value of ρ_1 at real frequencies. Equations (23) and (24) indicate that ρ_2 is

$$\rho_2 = \pm h \frac{\prod_{j=1}^k (s-s_{rj}) \prod_{i=1}^{n-k} (s+s_{oi}) (s+\sigma)^2}{\prod_{i=1}^n (s-s_{pi}) (s+\sigma)^2} . \quad (75)$$

The left-half s-plane zeros of ρ_1 change sign to become the right-half s-plane zeros of ρ_2 , etc.

If the numerator and denominator of ρ_1 are each multiplied by $\prod_{j=1}^k (s+s_{rj})$ and the numerator and denominator of ρ_2 are multiplied by $\prod_{i=1}^{n-k} (s-s_{oi})$, the return loss expansions for $\ln(1/\rho_1)$ and $\ln(1/\rho_2)$ can be rearranged as

$$\ln \frac{1}{\rho_1} = \ln \left[\frac{1}{\pm h} \left(\frac{\prod_{i=1}^n (s-s_{pi})}{\prod_{i=1}^{n-k} (s-s_{oi}) \prod_{j=1}^k (s-s_{rj})} \right) \right] + \quad (76)$$

$$\ln \left[\frac{\prod_{j=1}^k (s+s_{rj})}{\prod_{j=1}^k (s-s_{rj})} \right] + \ln \frac{(s+\sigma)^2}{(s-\sigma)^2} + \left\{ \begin{matrix} j\pi \\ \text{or} \\ 0 \end{matrix} \right\} ,$$

$$\ln \frac{1}{\rho_2} = \ln \left[\frac{1}{\pm h} \left(\frac{\prod_{i=1}^n (s-s_{pi})}{\prod_{i=1}^{n-k} (s-s_{oi}) \prod_{j=1}^k (s-s_{rj})} \right) \right] + \quad (77)$$

$$\ln \left[\frac{\prod_{i=1}^{n-k} (s-s_{oi})}{\prod_{i=1}^{n-k} (s+s_{oi})} \right] + \ln \frac{(s+\sigma)^2}{(s-\sigma)^2} + \left\{ \begin{matrix} j\pi \\ \text{or} \\ 0 \end{matrix} \right\} .$$

The first bracketed expressions in equations (76) and (77) are alike and have no right-half s-plane zeros. The first bracketed expressions are adjusted in sign to be non-negative at the origin, accounting for

the compensating factor of $j\pi$ or zero. The logarithm of the first bracketed expression is $\ln(1/|q_0|)$, in accordance with the procedure of Appendix D.

The contour integration for transmission zeros at infinity, as outlined in Appendix D, leads to

$$\int_0^\infty \ln \frac{1}{|q_1|} d\omega = \frac{\pi}{2} \left(\overset{(1)}{A}_1^\infty - 2 \sum_1^k S_{rj} - 4\sigma \right), \quad (78)$$

$$\int_0^\infty \ln \frac{1}{|q_1|} d\omega = \frac{\pi}{2} \left(\overset{(2)}{A}_1^\infty - 2 \sum_1^{n-k} S_{oi} \right), \quad (79)$$

for the $\overset{(1)}{A}_1^\infty$ and $\overset{(2)}{A}_1^\infty$ coefficients. (The superscripts refer to the "1" or "2" ends of the network.) However,

$$\sum_1^{n-k} S_{oi} = \sum S_{Ti} - \sum_1^k S_{rj}, \quad (80)$$

where $\sum S_{Ti}$ is the summation of all the zeros of q_1 except the σ 's.

Equation (79) becomes

$$\int_0^\infty \ln \frac{1}{|q_1|} d\omega = \frac{\pi}{2} \left(\overset{(2)}{A}_1^\infty + 2 \sum S_{Ti} - 2 \sum_1^k S_{rj} \right). \quad (81)$$

Thus, equations (78) and (81) form a pair for describing the $\overset{(1)}{A}_1^\infty$ coefficients. Were it not for the double zero term 4σ in equation (78), the summation of the zeros of q_1 would be inexorably fixed as the difference between $\overset{(2)}{A}_1^\infty$ and $\overset{(1)}{A}_1^\infty$.

As shown in Appendix D, the general relationships between the $\overset{(1)}{A}_1^\infty$ coefficients and behavior of $\ln(1/|q_1|)$ at real frequencies are given

as

$$\int_0^{\infty} \omega^{2m-2} \ln \frac{1}{|e_1|} d\omega = \quad (82)$$

$$(-1)^{m+1} \frac{\pi}{2} \left(A_{2m-1}^{(1)} - \frac{2}{2m-1} \sum_1^k S_{rj}^{2m-1} - \frac{4}{2m-1} \sigma^{2m-1} \right),$$

$$\int_0^{\infty} \omega^{2m-2} \ln \frac{1}{|e_1|} d\omega = \quad (83)$$

$$(-1)^{m+1} \frac{\pi}{2} \left(A_{2m-1}^{(2)} + \frac{2}{2m-1} \sum S_{Ti}^{2m-1} - \frac{2}{2m-1} \sum_1^k S_{rj}^{2m-1} \right),$$

where $m=1, 2, 3, \dots$. Alternate forms are

$$A_{2m-1}^{(1)} = (-1)^{m+1} \frac{2}{\pi} \int_0^{\infty} \omega^{2m-2} \ln \frac{1}{|e_1|} d\omega + \quad (84)$$

$$\frac{2}{2m-1} \sum_1^k S_{rj}^{2m-1} + \frac{4}{2m-1} \sigma^{2m-1},$$

$$A_{2m-1}^{(2)} = (-1)^{m+1} \frac{2}{\pi} \int_0^{\infty} \omega^{2m-2} \ln \frac{1}{|e_1|} d\omega + \quad (85)$$

$$\frac{2}{2m-1} \sum S_{Ti}^{2m-1} + \frac{2}{2m-1} \sum_1^k S_{rj}^{2m-1},$$

where $m=1, 2, 3, \dots$. Equations (84) and (85) relate the behavior of e_1 at real frequencies to the A^{∞} coefficients.

Simultaneous solutions of equations from integrals.--If network N' has transmission zeros of multiplicity n' at infinity and the network N'' has transmission zeros of multiplicity n'' at infinity, there are n' equations (82) and n'' equations (83) which satisfy the necessary

conditions on ρ_1 and ρ_2 . Should the desired network response over the entire real frequency range be known by the specification of $|\rho_1|$ (and, of course, $\ln(1/|\rho_1|)$) over that range, the gain integrals of equations (82) and (83) assume numerical values. The A^∞ 's, although very necessarily part of the description of the overall behavior of the entire network, are determined from the given load and source networks. It is entirely possible that a consistent set of simultaneous equations cannot be obtained with the integrals and the A^∞ 's alone. The s_{rj} 's are available for adding terms to bring about consistency in the simultaneous equations. The σ is available to adjust the tolerance, as is explained later in this chapter. In this regard, it should be noted that the addition of an s_{rj} term affects both equations (82) and (83), whereas the addition of the σ term affects only equation (82).

Limitations on Gain and Tolerance

For a particular network, there exists a value for the gain integral which cannot be exceeded. From equations (78) and (79), the value of the gain integral cannot exceed $(\pi/2)A_1^{(1)\infty}$ or $(\pi/2)A_1^{(2)\infty}$, whichever is smaller. (For convenience, in this study $A_1^{(2)\infty} \geq A_1^{(1)\infty}$.) Any right-half s-plane zeros of ρ_1 , whether they be simple or multiple, reduce the value of the gain integral.

As was stated in Chapter I, the optimum matching condition exists when $\ln(1/|\rho_1|)$ is constant in the pass band and zero elsewhere. This response is shown in Fig. 10 for a pass band extending from zero to ω_c . Fig. 10 shows a theoretical maximum value for a constant $\ln(1/|\rho_1|)$ and a given ω_c which can be obtained only through a system of networks having an infinite number of properly chosen elements.

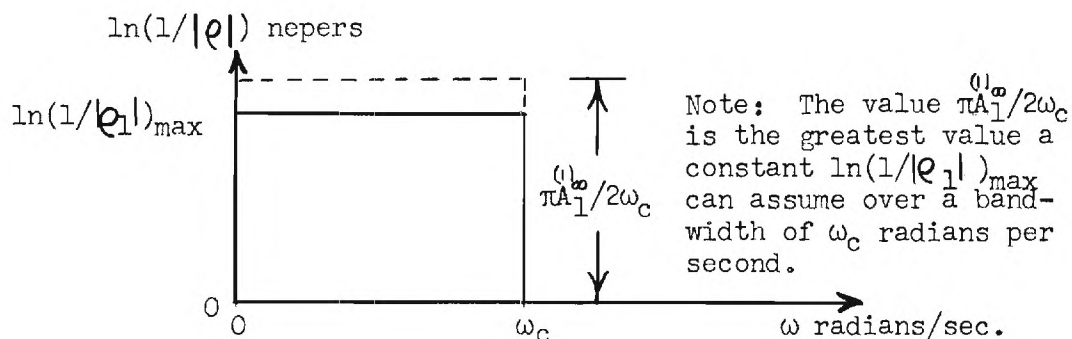


Fig. 10. Response of Ideal Low-pass Network

The networks of this study do not permit complete freedom of choice of circuit elements, since the elements of the source network are specified in addition to the elements of the load network.

It is apparent that some sort of systematic method should be employed to describe the behavior of a network with a finite number of elements and with elements specified at each end (load and source) of the network system. The Tchebycheff function of the first kind (24) leads to $\ln(1/|e_1|)$ which approximates a straight line in the pass band by oscillating between two given values in the pass band and approximates zero in the stop band by monotonically approaching zero. The Tchebycheff method is outlined in the following section.

Development of Tchebycheff method.—The Tchebycheff behavior is obtained by expressing the transmission as

$$|t|_{s=j\omega}^2 = \frac{1}{(1 + k^2) + \epsilon^2 T_n^2(\omega/\omega_c)}, \quad (86)$$

where k and ϵ are real constants, $T_n(\omega/\omega_c)$ is the Tchebycheff function of the first kind, order n , argument ω/ω_c , and ω_c is the upper radian

frequency limit of the pass band at the point where the oscillatory behavior changes to monotonic behavior. The Tchebycheff function

$T_n(\omega/\omega_c)$ is defined as $\cos(n \cos^{-1}(\omega/\omega_c))$.

Since

$$|q_1|_{s=j\omega}^2 = 1 - |t|_{s=j\omega}^2 = [q_1(s)q_1(-s)]_{s=j\omega}, \quad (87)$$

equations (86) and (87) lead to

$$|q_1|_{s=j\omega}^2 = \frac{\left[\left(\frac{k^2}{\epsilon^2} \right) + T_n^2 \left(-j \frac{s}{\omega_c} \right) \right]}{\left[\left(\frac{1+k^2}{\epsilon^2} \right) + T_n^2 \left(-j \frac{s}{\omega_c} \right) \right]}_{s=j\omega}. \quad (88)$$

In the range from ω equal to zero to ω_c , $T_n^2(\omega/\omega_c)$ oscillates between zero and one. In the range from ω equal to ω_c to infinity, $T_n^2(\omega/\omega_c)$ monotonically increases from one to infinity. Sketches of $|t|^2$, $|q_1|^2$, and $\ln(1/|q_1|)$ for n equal to three are shown in Fig. 11.

The left-half s -plane poles and zeros of q_1 can be determined from the left half s -plane poles and zeros of

$$q_1(s)q_1(-s) = \frac{\left[\left(\frac{k^2}{\epsilon^2} \right) + T_n^2 \left(-j \frac{s}{\omega_c} \right) \right]}{\left[\left(\frac{1+k^2}{\epsilon^2} \right) + T_n^2 \left(-j \frac{s}{\omega_c} \right) \right]}, \quad (89)$$

which is equation (88) extended into the complex s -plane. There cannot be, of course, any right-half s -plane poles of q_1 . Any right-half s -plane zeros can be added to q_1 later, if required.

From the denominator of the right-hand side of equation (89), poles of q_1 occur when

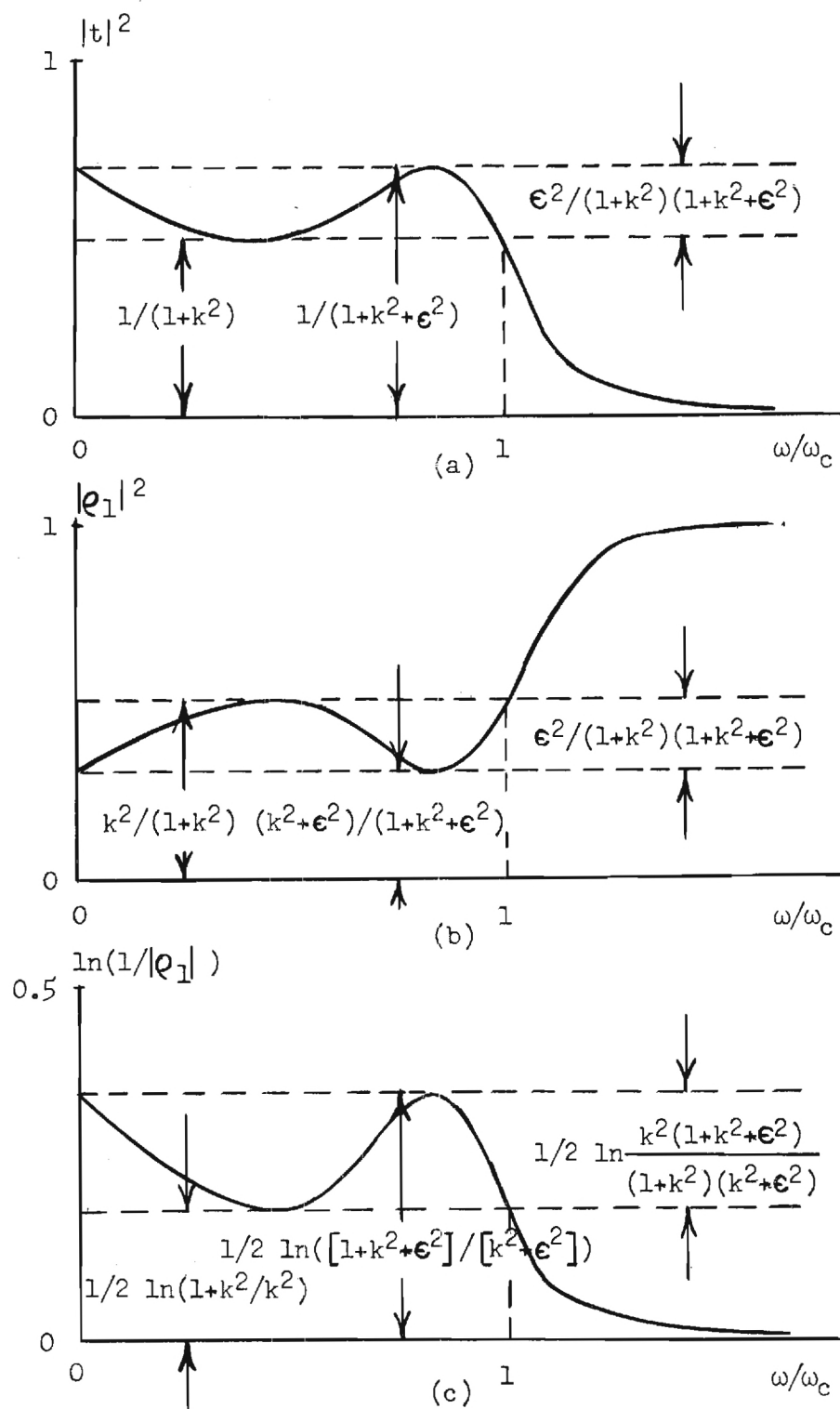


Fig. 11. Variations of $|t|^2$, $|e_1|^2$, and $\ln(1/|e_1|)$ with ω/ω_c

$$T_n\left(-j \frac{s}{\omega_c}\right) = \pm j \sqrt{\frac{1+k^2}{\epsilon^2}}, \quad (90)$$

or

$$\cos\left(n \cos^{-1} \frac{s}{j\omega_c}\right) = \pm jc, \quad (91)$$

where c is substituted for the radical of equation (90). The term $[\cos^{-1}(s/j\omega_c)]$ can be complex and is replaced by $(\alpha - j\beta)$,

$$\begin{aligned} \cos(n\alpha - jn\beta) &= \cos n\alpha \cosh n\beta + \\ j \sin n\alpha \sinh n\beta &= \pm jc. \end{aligned} \quad (92)$$

Equation (92) yields

$$\cos n\alpha \cosh n\beta = 0, \quad (93)$$

$$\sin n\alpha \sinh n\beta = \pm jc. \quad (94)$$

Equation (93) is satisfied when $n\alpha$ equals an odd multiple of $\pi/2$, or

$$\alpha = \frac{\pi}{2n} (2m-1), \quad (95)$$

where $m=1, 2, 3, \dots, n$. When equation (95) is substituted in equation (94), the result is

$$\beta = \pm \frac{1}{n} \sinh^{-1} c. \quad (96)$$

From equations (95) and (96), the general expression for a pole of

equation (89) is

$$s_{pi} = \mp \omega_c \left[\left\{ \sin \frac{\pi}{2n} (2m-1) \right\} \left\{ \sinh \left(\frac{1}{n} \sinh^{-1} c \right) \right\} + \right. \quad (97)$$

$$\left. j \left\{ \cos \frac{\pi}{2n} (2m-1) \right\} \left\{ \cosh \left(\frac{1}{n} \sinh^{-1} c \right) \right\} \right],$$

where $m=1, 2, 3, \dots, n$.

It should be noted that s_{pi} in equation (97) is in the form $\sigma + j\omega$, where σ is measured along the real axis and $j\omega$ is measured along the imaginary axis in the s -plane. Equation (97) indicates that the s_{pi} 's lie on an ellipse centered at the origin and having major and minor axes of $\omega_c \cosh([\sinh^{-1}c]/n)$ and $\omega_c \sinh([\sinh^{-1}c]/n)$, respectively. The equation of the ellipse is

$$\frac{\omega_c^2 \left\{ \sin^2 \frac{\pi}{2n} (2m-1) \right\} \left\{ \sinh^2 \left(\frac{1}{n} \sinh^{-1} c \right) \right\}}{\omega_c^2 \sinh^2 \left(\frac{1}{n} \sinh^{-1} c \right)} + \quad (98)$$

$$\frac{\omega_c^2 \left\{ \sin^2 \frac{\pi}{2n} (2m-1) \right\} \left\{ \cosh^2 \left(\frac{1}{n} \sinh^{-1} c \right) \right\}}{\omega_c^2 \cosh^2 \left(\frac{1}{n} \sinh^{-1} c \right)} = 1.$$

As a further reduction, if c is replaced by na , or

$$na = \sinh^{-1} \sqrt{\frac{1+k^2}{\epsilon^2}}, \quad \sinh na = \sqrt{\frac{1+k^2}{\epsilon^2}}, \quad (99)$$

the axes of the ellipse can be expressed as $\omega_c \cosh a$ and $\omega_c \sinh a$.

The same process can be repeated to find the left-half s-plane zeros of $Q_1(s)Q_1(-s)$. In this case, nb is set equal to $\sinh^{-1}\sqrt{k^2/\epsilon^2}$, and the axes of the ellipse for the zeros are given by $\omega_c \cosh b$ and $\omega_c \cosh a$.

The poles and zeros of $Q_1(s)Q_1(-s)$ are summarized as

$$S_{pi} = \mp \omega_c \left\{ \sin \frac{\pi}{2n} (2m-1) \right\} \{ \sinh a \} + \quad (100)$$

$$j \left\{ \cos \frac{\pi}{2n} (2m-1) \right\} \{ \cosh a \},$$

$$S_{oi} = \mp \omega_c \left\{ \sin \frac{\pi}{2n} (2m-1) \right\} \{ \sinh b \} + \quad (101)$$

$$j \left\{ \cos \frac{\pi}{2n} (2m-1) \right\} \{ \cosh b \}.$$

There are a total of $2n$ distinct poles (or zeros), of which n are in the left-half s-plane and n are in the right-half s-plane. The minus signs lead to left-half s-plane poles and zeros. Thus, n is the total number of transmission zeros of the complete system at infinity, as indicated by the relationship between Q_1 and t .

In any particular problem, a certain number of $A_{\infty}^{(1)}$ and $A_{\infty}^{(2)}$ coefficients are specified by the given networks N' and N'' . For example, if N' were such that t' has a second order zero at infinity, $A_{\infty}^{(1)}$ and $A_{\infty}^{(2)}$ would be fixed by the circuit elements of N' . Since the overall $\ln(1/|Q_1|)$ is to exhibit Tchebycheff behavior, the $A_{\infty}^{(1)}$ and $A_{\infty}^{(2)}$ coefficients appear in simultaneous equations as part of the description of the Tchebycheff behavior. Right-half s-plane zeros of $Q_1(s)$ are

introduced where necessary to obtain consistent simultaneous equations or to adjust the tolerance. It must be remembered, however, that a matching left-half s-plane pole is added for every added right-half s-plane zero. Although the addition of a simple or double right-half s-plane zero of ρ_1 results in the addition of a zero of transmission, the total number of zeros of transmission at infinity is not changed, since the added zero of transmission is in the right-half s-plane.

By using closed forms of trigonometric series (25) in conjunction with equations (100) and (101), it is possible to determine

$$\sum_{i=1}^n S_{pi} = -\omega_c \frac{\sinh a}{\sin \pi/2n}, \quad (102)$$

$$\sum_{i=1}^n S_{oi} = -\omega_c \frac{\sinh b}{\sin \pi/2n}. \quad (103)$$

The s_{pi} and s_{oi} of equations (102) and (103) are the left-half s-plane poles and zeros of ρ_1 . They can also be considered as all the poles and zeros of ρ_0 , which is ρ_1 less any right-half s-plane zeros and matching left-half s-plane poles. When equations (102) and (103) are combined as shown in equation (D-3) of Appendix D, the result is

$$F_1^\infty = \frac{1}{1} \left(\sum_{i=1}^n S_{oi} - \sum_{i=1}^n S_{pi} \right) = \omega_c \frac{\sinh a - \sinh b}{\sin \pi/2n}. \quad (104)$$

The F_1^∞ term is the first coefficient of the Taylor series expansion for $\ln(1/\rho_0)$ as given by equation (D-2). Further, it is shown in Appendix D that

$$F_1^\infty = \frac{2}{\pi} \int_0^\infty \ln \frac{1}{|e|} d\omega. \quad (105)$$

Reference to equation (84) indicates that

$$\overset{(1)}{A}_1^\infty = \omega_c \frac{\sinh a - \sinh b}{\sin \pi/2n} + 2 \sum_1^k S_{rj} + 4\sigma, \quad (106)$$

where there are k simple right-half s -plane zeros of ϱ_1 and one double right-half s -plane zero of ϱ_1 at $s=\sigma$. Through use of similar methods, the expression for $\overset{(1)}{A}_3^\infty$ becomes

$$\overset{(1)}{A}_3^\infty = -\frac{\omega_c^3}{4} \left[\frac{\sinh 3a - \sinh 3b}{3 \sin 3\pi/2n} + \frac{\sinh a - \sinh b}{\sin \pi/2n} \right] + \quad (107)$$

$$\frac{2}{3} \sum S_{rj}^3 + \frac{4}{3} \sigma^3.$$

The expressions for $\overset{(1)}{A}_5^\infty$, etc., are developed in the same manner as are the expressions for $\overset{(1)}{A}_1^\infty$ and $\overset{(1)}{A}_3^\infty$.

Whenever simple right-half s -plane zeros are added to ϱ_1 in such a way as to keep $|\varrho_1|$ unchanged at real frequencies, equation (39) points out that compensating zeros are added in the left-half s -plane. The result is that the summation of the zeros or the summation of powers of the zeros of ϱ_1 are unchanged by the addition of simple right-half s -plane zeros to ϱ_1 . In equation form, this is

$$\sum S_{Ti}^{2m-1} = \sum_1^n S_{oi}^{2m-1}, \quad (108)$$

where $\sum S_{Ti}^{2m-1}$ is the summation of the $(2m-1)$ th powers of all the zeros of ϱ_1 , regardless of location in the s -plane, and $\sum_1^n S_{oi}^{2m-1}$ is the summation of

the $(2m-1)$ th powers of the left-half s-plane zeros of reflection as determined from equation (101). The index m assumes values 1, 2, 3, - - -.

When the right-hand side of equation (108) is substituted in equation (85) and $\sum_{oi}^n s_{oi}^{2m-1}$ is replaced by its hyperbolic form (see equation (103) for the hyperbolic form for $m=1$), the values of successive $A_\infty^{(2)}$'s become

$$A_1^{(2)} = \omega_c \frac{\sinh a + \sinh b}{\sin \pi/2n} + 2 \sum_1^k s_{rj}, \quad (109)$$

$$A_3^{(2)} = \frac{\omega_c^3}{4} \left[\frac{\sinh 3a + \sinh 3b}{\sin 3\pi/2n} + \frac{\sinh a + \sinh b}{\sin \pi/2n} \right] + \frac{2}{3} \sum_1^k s_{rj}^3. \quad (110)$$

Discussion of maximizing.---With a Tchebycheff response, the question arises as to whether it is possible to secure a maximum transmission behavior. For example, if the load and source networks are specified, is it always possible to find a matching network which optimizes the transmission by producing a minimum in $|Q_1|_{\max}$? Secondly, are there conditions under which the variation of the transmission in the pass band is least?

The answer to the first question is found by attempting to minimize $|Q_1|_{\max}$ for the simplest of all cases -- single elements in the load and source networks. The simplest case is chosen because the least number of elements of the overall network is fixed, leaving the greatest latitude for choice of the remaining elements.

A comparison between equations (106) and (109) reveals that an increase in the $\sum_1^k s_{rj}$ term always decreases a , but does not change b . An increase in σ always decreases a while simultaneously increasing b .

Since the expression for $|Q_1|_{\max}$ is (as listed in Fig. 11)

$$|Q_1|_{\max} = \sqrt{\frac{k^2 + \epsilon^2}{1 + k^2 + \epsilon^2}} = \frac{\cosh nb}{\cosh na}, \quad (111)$$

it is evident that an increase in either $\sum_{j=1}^k s_{rj}$ or σ , or both, always increases $|Q_1|_{\max}$. Thus, the minimum value of $|Q_1|_{\max}$ is found by substituting in equation (111) the a and b values of

$$A_1^{(1)\infty} = \omega_c \frac{\sinh a - \sinh b}{\sin \pi/2n}, \quad (112)$$

$$A_1^{(2)\infty} = \omega_c \frac{\sinh a + \sinh b}{\sin \pi/2n}. \quad (113)$$

If the A_1^∞ 's, ω_c , and n are fixed by the data of the problem, the minimum value of $|Q_1|_{\max}$ for Tchebycheff behavior is unique.

Effects of imbedded all pass sections.--The second question can be answered by considering the effect of the parameter σ , which is added when an all pass structure is imbedded in the matching network. The purpose of σ is to provide an added term in the gain integral equations. The added term is present in one set of equations, but it does not appear in the other. The σ term supplies the parameter needed to free an otherwise fixed difference between corresponding $A_1^{(1)\infty}$ and $A_1^{(2)\infty}$ coefficients. However, the added σ term can be used to adjust the ripple or tolerance of $|Q_1|$ (as well as that of $|t|$ and that of $\ln(1/|Q_1|)$) in the pass band. It is pointed out that the possibilities of the σ term have not been explored by other investigators, doubtless because the need for such a term is not apparent until matching

networks are interpolated between two arbitrary networks. The manner in which the σ term affects the gain and tolerance of the entire network system is included in the following section.

Variation of parameters and effect on network response.--In order to gain a comprehensive view of the manner in which σ and other parameters affect the network behavior, it is instructive to study a range of possible choices for the parameters. For any given problem with source and load networks of arbitrary complexity, the optimum behavior cannot exceed the behavior of a system with single-element source and load networks if those single elements are identical with the exterior elements of the original source and load networks. Thus, the simplest case is used for comparison of parameter effects.

The ratio

$$\frac{\tanh b}{\tanh a} = \frac{|e|_{\min}}{|e|_{\max}} = \delta \quad (114)$$

is chosen as a measure of tolerance or ripple in the pass band. When δ equals unity, the ripple or tolerance is zero. If $A_1^{(2)\infty}$ is replaced by $HA_1^{(1)\infty}$, where H is a real constant equal to or greater than unity, the companion equations for the A_1^∞ 's are

$$A_1^{(1)\infty} = \omega_c \frac{\sinh a - \sinh b}{\sin \pi/2n} + 4\sigma = \quad (115)$$

$$\omega_c \frac{\sinh a - \sinh b}{\sin \pi/2n} + 4DA_1^{(1)\infty},$$

$$A_1^{(2)\infty} = HA_1^{(1)\infty} = \omega_c \frac{\sinh a + \sinh b}{\sin \pi/2n}, \quad (116)$$

where D in equation (115) is equal to $\sigma/A_1^{(1)\infty}$ and is given by $0 \leq D \leq 0.25$.

The parameter D permits the behavior of σ to be normalized with respect to $A_1^{(1)\infty}$.

As D increases from zero to one quarter, the numerator of equation (114) increases monotonically, while the denominator of equation (114) decreases monotonically. Thus, δ increases monotonically as D increases from zero to one quarter. This is the answer to the second question above regarding the variation of transmission in the pass band. In the event $D=0.25$, a is equal to b , and the minimum and maximum values of $|Q_1|$ are unity, indicating that while there is no ripple, there is no transmission either.

Combination of equations (115) and (116) yields

$$\sinh b = \left(\sin \frac{\pi}{2n} \right) \left(\frac{A_1^{(1)\infty}}{2\omega_c} [H - 1 + 4D] \right), \quad (117)$$

$$\sinh a = \left(\sin \frac{\pi}{2n} \right) \left(\frac{A_1^{(1)\infty}}{2\omega_c} [H + 1 - 4D] \right). \quad (118)$$

Thus δ becomes

$$\delta = \frac{\tanh n \left\langle \sinh^{-1} \left\{ \left(\sin \frac{\pi}{2n} \right) \left(\frac{A_1^{(1)\infty}}{2\omega_c} [H - 1 + 4D] \right) \right\} \right\rangle}{\tanh n \left\langle \sinh^{-1} \left\{ \left(\sin \frac{\pi}{2n} \right) \left(\frac{A_1^{(1)\infty}}{2\omega_c} [H + 1 - 4D] \right) \right\} \right\rangle}. \quad (119)$$

Equation (119) yields pertinent information regarding the behavior of the pass band ripple of the reflection and transmission coefficients. For example, if $A_1^{(1)\infty}$, H , D , and ω_c are fixed and n approaches

infinity, equation (119) becomes

$$\lim_{n \rightarrow \infty} \delta = \frac{\tanh \left\langle \left(\frac{\pi}{2} \right) \left(\frac{A_1^{(1)\infty}}{2\omega_c} \right) (H - 1 + 4D) \right\rangle}{\tanh \left\langle \left(\frac{\pi}{2} \right) \left(\frac{A_1^{(1)\infty}}{2\omega_c} \right) (H + 1 - 4D) \right\rangle}. \quad (120)$$

For a fixed H and a fixed $D < 0.25$, it is interesting to note that $0 < \delta < 1$. This indicates that a finite difference can exist between the upper and lower ripple limits even when the number of elements in the matching network approaches infinity.

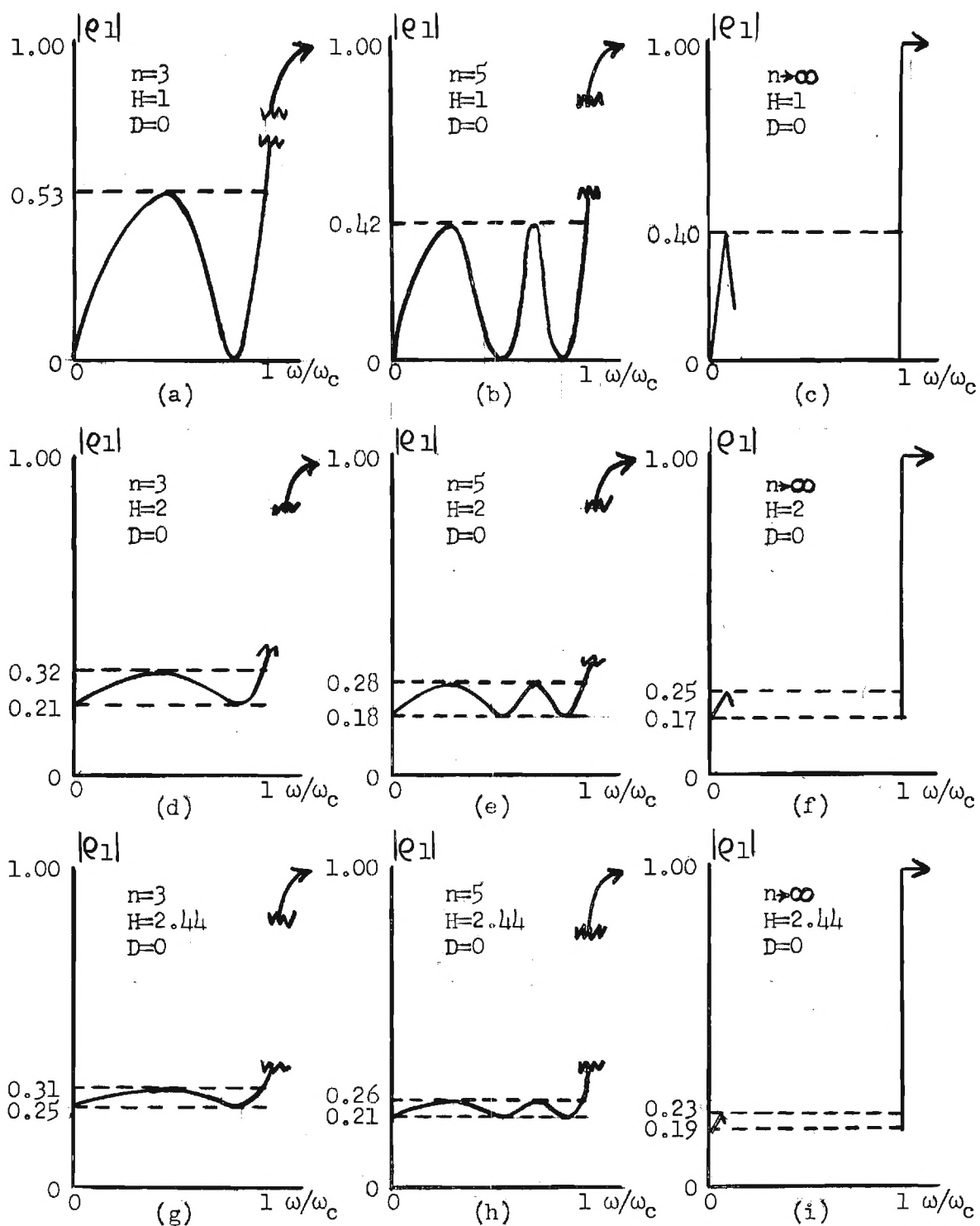
The upper and lower values of $|e_1|$ as given in Fig. 11 can be rearranged in the above case for $n \rightarrow \infty$ as

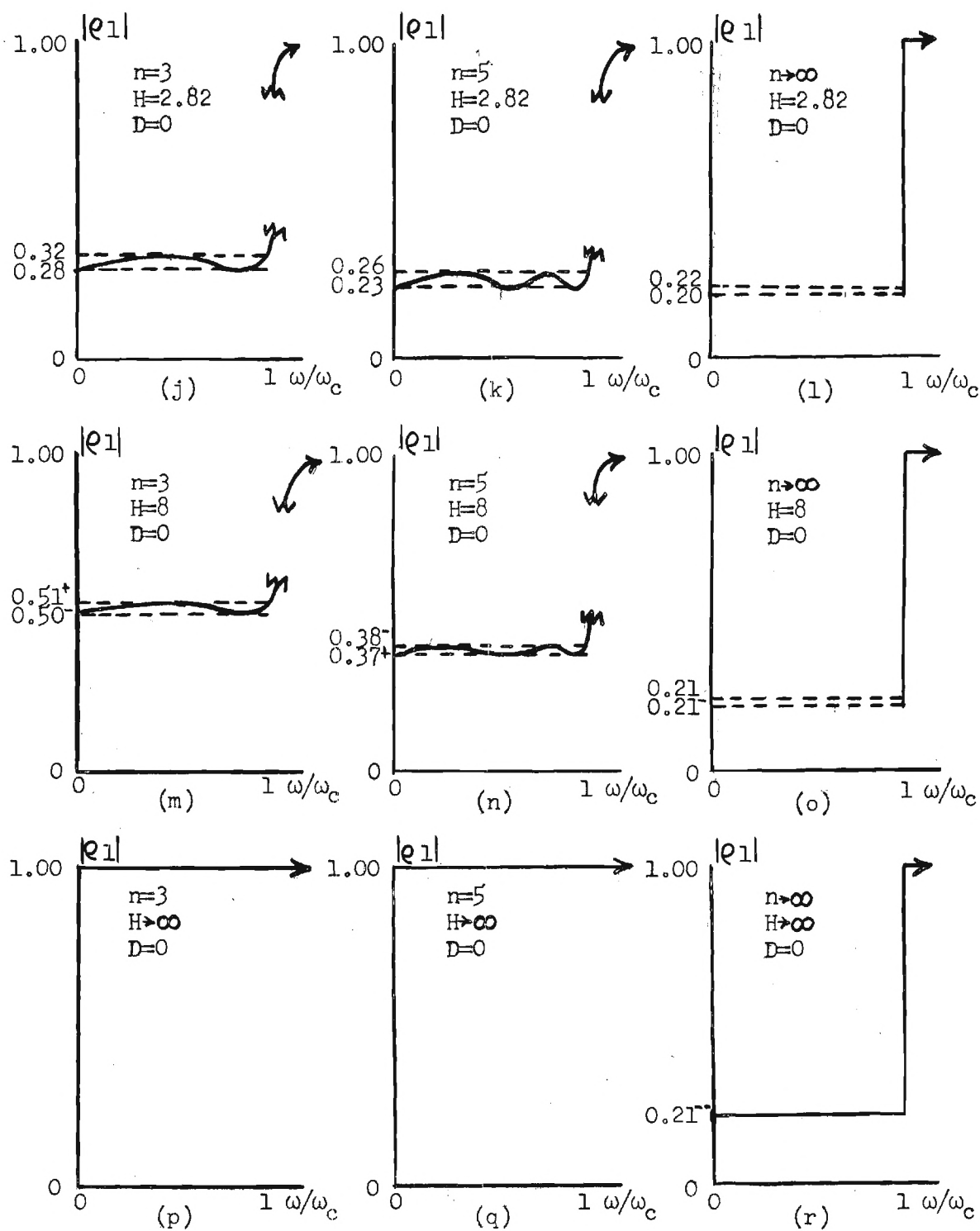
$$\lim_{n \rightarrow \infty} |e_1|_{\max} = \frac{\cosh \left\langle \left(\frac{\pi}{2} \right) \left(\frac{A_1^{(1)\infty}}{2\omega_c} \right) (H - 1 + 4D) \right\rangle}{\cosh \left\langle \left(\frac{\pi}{2} \right) \left(\frac{A_1^{(1)\infty}}{2\omega_c} \right) (H + 1 - 4D) \right\rangle}, \quad (121)$$

$$\lim_{n \rightarrow \infty} |e_1|_{\min} = \frac{\sinh \left\langle \left(\frac{\pi}{2} \right) \left(\frac{A_1^{(1)\infty}}{2\omega_c} \right) (H - 1 + 4D) \right\rangle}{\sinh \left\langle \left(\frac{\pi}{2} \right) \left(\frac{A_1^{(1)\infty}}{2\omega_c} \right) (H + 1 - 4D) \right\rangle}. \quad (122)$$

The variations of $|e_1|_{\max}$, $|e_1|_{\min}$, and δ are shown for various H 's with D fixed at zero in Fig. 12(a), (b), (c), or any other horizontal series of sketches of Fig. 12. (It is emphasized that H , $A_1^{(1)\infty}$, n , ω_c , etc., are fixed quantities for a particular problem.)

If, for the sake of argument, it were possible to vary the element of the source network, H could assume all positive values from

Fig. 12. Variation of $|e_1|$ with n and H (Sheet 1)

Fig. 12. Variation of $|e_1|$ with n and H (Sheet 2)

unity on toward infinity. As H increases from unity toward infinity with D , $A_1^{(1)\infty}$, n , ω_c fixed (see equation (119)), δ increases monotonically from an initial value toward unity. On the other hand, the value of $|Q_1|_{\max}$ decreases to a minimum and then increases toward unity as H increases from unity toward infinity. The $HA_1^{(1)\infty}$ or $A_1^{(2)\infty}$ term for minimum $|Q_1|_{\max}$ is identically the $A_1^{(2)\infty}$ term which would be found by the Fano procedure (26). The above effect is shown in Fig. 12(a), (d), (g), (j), (m), (p), or any other vertical series of sketches of Fig. 12.

As first n and then H are allowed to approach infinity with $A_1^{(1)\infty}$ and ω_c fixed and D equal to zero, equation (120) indicates that δ approaches unity, and the ripple approaches zero. Accordingly, $|Q_1|_{\max}$ is equal to $|Q_1|_{\min}$ throughout the pass band. The value of $|Q_1|_{\max}$ is found by letting H approach infinity in the exponential form of equation (121). The result is

$$\lim_{H \rightarrow \infty} \left[\lim_{n \rightarrow \infty} |Q_1|_{\max} \right] = \frac{1}{e^{\pi A_1^{(1)\infty} / 2\omega_c}}, \quad (123)$$

which is shown in Fig. 12(r). It is interesting to note that equation (123) represents the ultimate in reflection coefficient magnitudes for the type of network under consideration. When the constant $\ln(1/|Q_1|_{\max})$ taken from equation (123) is integrated between zero and ω_c , the result is

$$\int_0^{\omega_c} \ln(e^{\pi A_1^{(1)\infty} / 2\omega_c}) d\omega = \frac{\pi}{2} A_1^{(1)\infty}, \quad (124)$$

which is the greatest value the gain integral of equation (124) can

ever attain, with the chosen $A_1^{(1)\infty}$. It is pointed out again that the result of equation (124) can never be reached so long as $HA_1^{(1)\infty}$ (or $A_1^{(2)\infty}$) is finite.

An interesting special case occurs when H is equal to unity. Since $A_1^{(2)\infty}$ is thereby equal to $A_1^{(1)\infty}$, the first elements of the source and load networks are numerically equal. The value for $|Q_1|_{\min}$ is given by $[\sinh nb / \sinh na]$ (see Fig. 11), which can be expressed as

$$|Q_1|_{\min} = \frac{\sinh nb}{\sinh na} = \quad (125)$$

$$\frac{\sinh n \left\langle \sinh^{-1} \left\{ \sin \frac{\pi}{2n} \left(\frac{A_1^{(1)\infty}}{2\omega_c} \right) (H - 1 + 4D) \right\} \right\rangle}{\sinh n \left\langle \sinh^{-1} \left\{ \sin \frac{\pi}{2n} \left(\frac{A_1^{(1)\infty}}{2\omega_c} \right) (H + 1 - 4D) \right\} \right\rangle}$$

with the aid of equations (117) and (118). If D is zero, equation (125) indicates that $|Q_1|_{\min}$ is equal to zero for any value of n . Fig. 12(a), (b), (c) show the variation of $|Q_1|$ with ω and n for H equal to unity and D equal to zero. The above situation is to be avoided in matching networks, since perfect matching at some frequencies in the pass band leads to inefficient use of the area under the gain curve. As can be seen from equation (125), the introduction of D serves to increase $|Q_1|_{\min}$ and allows $\ln(1/|Q_1|)$ to vary between two non-zero values in the pass band.

As D varies from zero to 0.25 with H and n fixed, equation (119) indicates that δ approaches unity for each value of the ratio $A_1^{(1)\infty}/\omega_c$. This behavior is shown in Fig. 13(a), (d), (g), or any other vertical series of sketches of Fig. 13. It was shown previously in equation

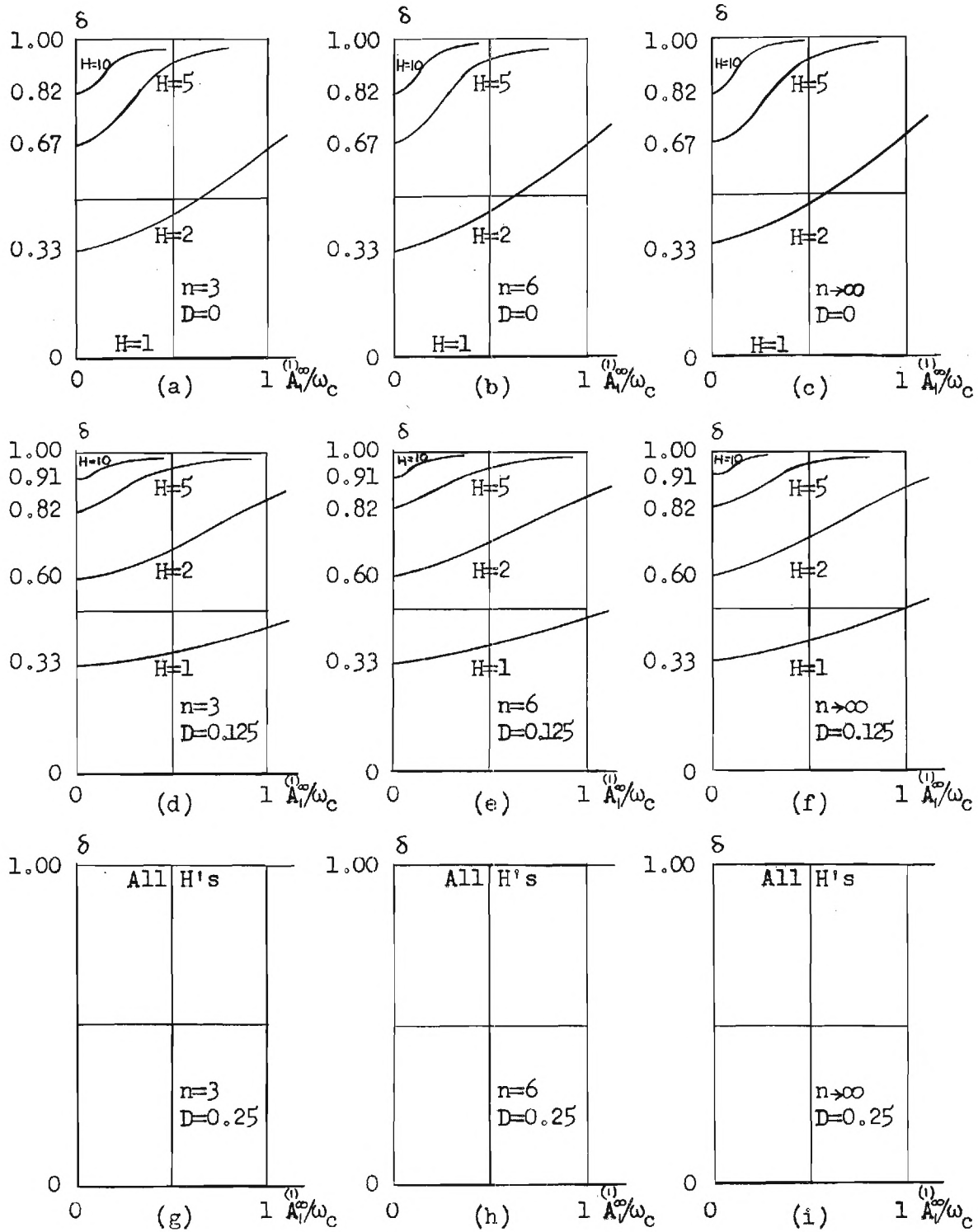


Fig. 13. Variation of δ vs. $A_1^{(1)}/\omega_c$ for Various n , D , and H

(115) that the gain vanishes when D is equal to 0.25.

As $n \rightarrow \infty$ with D and H fixed, equation (119) and (120) indicate that the δ values change only slightly for a particular value of $A_1^{(0)}/\omega_c$. This behavior is shown in Fig. 13(a), (b), (c), or any other horizontal series of sketches of Fig. 13.

As $A_1^{(0)}/\omega_c$ approaches zero with D and H fixed, equation (119) reduces to

$$\lim_{A_1^{(0)}/\omega_c \rightarrow 0} \delta = \frac{H - 1 + 4D}{H + 1 - 4D}, \quad (126)$$

regardless of n . This behavior is shown in Fig. 13(a), (b), (c), or any other horizontal series of sketches of Fig. 13.

Optimum Gain Considerations

It is noted earlier that simple zeros of reflection of ρ_1 in the right-half s -plane often are required for obtaining consistency in the simultaneous gain integral equations. This section offers some suggestions as to the manner of choosing the simple zeros of reflection. The double zeros of reflection are assumed to be zero for this discussion.

When the pass band extends from zero to ω_c , and when $\ln(1/|\rho_1|)$ is assumed to be constant over the pass band and zero in the stop band, equations of the form given by equation (82) can be integrated from zero to ω_c and arranged as

$$\left[\left(\ln \frac{1}{|\rho_1|} \right) \left(\frac{2}{\pi} \omega_c \right) \right] = A_1^{(0)} - 2 \sum_1^k S_{rj}, \quad (127)$$

$$\left[\left(\ln \frac{1}{|q_1|} \right) \left(\frac{2}{\pi} \omega_c \right) \right] = -\frac{3}{\omega_c^2} A_3^{(1)\infty} + \frac{2}{3} \sum_1^k S_{rj}^3, \quad (128)$$

$$\left[\left(\ln \frac{1}{|q_1|} \right) \left(\frac{2}{\pi} \omega_c \right) \right] = \frac{5}{\omega_c^4} A_5^{(1)\infty} - \frac{2}{5} \sum_1^k S_{rj}^5, \quad (129)$$

$$\left[\left(\ln \frac{1}{|q_1|} \right) \left(\frac{2}{\pi} \omega_c \right) \right] = (-1)^{m+1} \left[\frac{2m-1}{\omega_c^{2m-2}} A_{2m-1}^{(1)\infty} - \frac{2}{2m-1} \sum_1^k S_{rj}^{2m-1} \right], \quad (130)$$

where $m=1, 2, 3, \dots$. The s_{rj} 's of the set of equations represented by equations (127) through (130) are the added simple right-half s -plane zeros of q_1 and the roots of the polynomial

$$(s-s_{r1})(s-s_{r2})\dots(s-s_{rk}). \quad (131)$$

For a specified value of m , it is desired to solve the system of equations (127) through (130) to find the number and location of the s_{rj} 's such that $[\ln(1/|q_1|)(2\omega_c/\pi)]$ assumes its greatest value. The form of the s_{rj} 's is of great importance in obtaining the largest gain integrals for practical cases.

No added right-half s -plane zeros of reflection.--When a zero of transmission at infinity of order one exists, equation (130) with m equal to unity forms the system of equations. It is obvious that $[\ln(1/|q_1|)(2\omega_c/\pi)]$ is greatest when there are no added right-half s -plane zeros; none is needed to bring about consistency.

Added simple right-half s -plane zero of reflection.--When a zero of transmission at infinity of order two exists, a set of two equations is formed from equation (130) with m equal to one and two. Equations

(127) and (128) are specific examples of the set of equations.

The s_{rj} 's must be chosen in such a manner as to make $\sum_{rj}^k s_{rj}$ as small as possible, consistent with as large a $\sum_{rj}^k s_{rj}^3$ as possible. It is stated and proved in the literature (27) that $\sum_{rj}^k s_{rj}$ should be a single positive quantity ϵ_1 , corresponding to one real, positive root. The value of ϵ_1 can be found by simultaneous solution of equations (127) and (128) when $\sum_{rj}^k s_{rj}$ and $\sum_{rj}^k s_{rj}^3$ are replaced by ϵ_1 and ϵ_1^3 . If the special case

$$A_1^{(1)\infty} = -\frac{3}{\omega_c} A_3^{(1)\infty} \quad (132)$$

is discounted, the significant fact emerges that the simplest of all possible added right-half s-plane zero combinations is also the optimum zero combination.

A comment concerning the utility of a degenerate zero is appropriate at this point. If the term $-(3A_3^{(1)\infty}/\omega_c^2)$ is greater than $A_1^{(1)\infty}$, then $A_3^{(1)\infty}$ must be negative, since $A_1^{(1)\infty}$ is positive. In Appendix C, it is stated that the presence of a degenerate zero when the zero of transmission is of multiplicity two increases the value of the $A_3^{(1)\infty}$ term. An increase in $A_3^{(1)\infty}$ thus reduces the value of $-(3A_3^{(1)\infty}/\omega_c^2)$. By the addition of the proper amount of shunt capacitance or series inductance (whichever is required) to the first section of network N'' , a zero of sufficient degeneracy between networks N' and N'' can be found to bring about the special case of equation (132). The term $[(\ln(1/|Q_1|))(2\omega_c/\pi)]$ assumes its greatest value because no added right-half s-plane zeros are needed. Two and more added simple right-half s-plane zeros of reflection.--When

the multiplicity of the zero of transmission at infinity is three, the m of equation (130) is also three. The set of equations are equations (127), (128), and (129). It is not possible to find an explicit expression for the type and location of the added zero(s), since the degree of some of the equations is equal to five. It is possible, however, to simulate a typical case and thereby arrive at a suggested method of procedure.

The following values are chosen for the typical case: $A_1^{(0)} = 4.0$, $A_3^{(0)} = 2.0$, and, for simplicity, $[(\ln(1/|Q_1|))(2\omega_c/\pi)]$ is replaced by g . The parameter g is found for various values of $A_3^{(0)}$ for (a) two complex roots, (b) one real and two complex roots, and (c) four complex roots. A comparison of the results for the various root systems shows that the two complex root case leads to the greatest value of g for all $A_3^{(0)}$ values except a small range of values of $A_3^{(0)}$. The method of reaching this conclusion follows.

Equations (127) through (129) can be reduced to

$$g = 4 - 2 \sum_{j=1}^k s_{rj}, \quad (133)$$

$$2g = 3 + \sum_{j=1}^k s_{rj}^3, \quad (134)$$

$$16g = 5A_3^{(0)} - 2 \sum_{j=1}^k s_{rj}^5. \quad (135)$$

The two complex roots can be represented as $Ae^{j\theta}$ and $Ae^{-j\theta}$, where A and θ are constants. The $\sum_{j=1}^k s_{rj}$ becomes $2A\cos\theta$, $\sum_{j=1}^k s_{rj}^3$ becomes $2A^3\cos3\theta$, and $\sum_{j=1}^k s_{rj}^5$ becomes $2A^5\cos5\theta$. A further simplification can be achieved through use of Tchebycheff polynomials of the first kind to replace the trigonometric quantities. This method is suggested in a recent Bureau

of Standards publication (28). The values selected for use here are listed by Klein (29).

$$\cos \theta = T_1(x) = x, \quad (136)$$

$$\cos 3\theta = T_3(x) = 4x^3 - 3x, \quad (137)$$

$$\cos 5\theta = T_5(x) = 16x^5 - 20x^3 + 5x. \quad (138)$$

Combination of equations (136) through (138) with equations (133) through (135) yields

$$g = 4 - 4(Ax), \quad (139)$$

$$2g = 3 + 8(Ax)^3 - 6(Ax)A^2, \quad (140)$$

$$16g = 5A_3^{(1)} - 64(Ax)^5 + 80(Ax)^3 - 20(Ax)A^4. \quad (141)$$

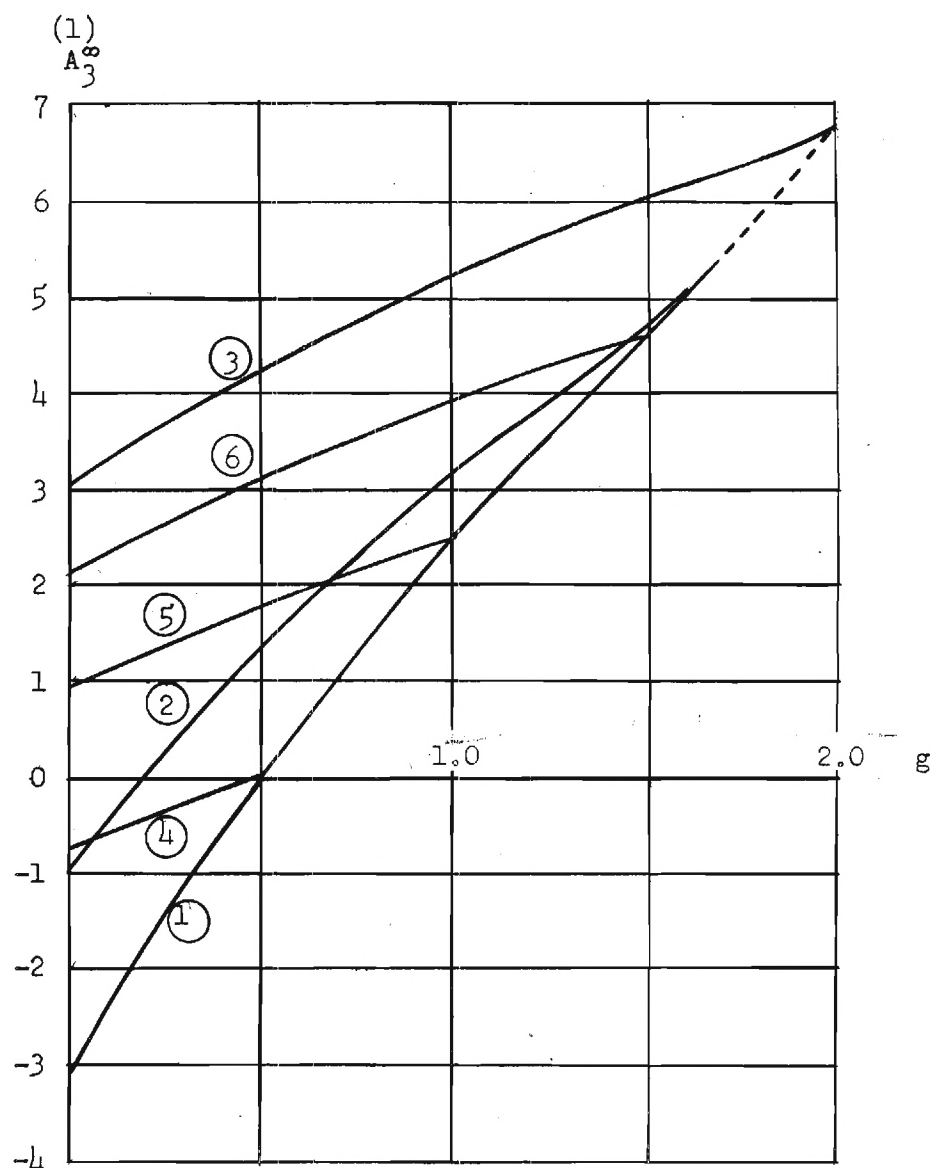
The greatest possible value which g can attain is two, as can be shown from a consideration of equations (139) and (140). If g assumes values in the range between zero and two, simultaneous solution of equations (139) through (141) yields a value of $A_3^{(1)}$ for each value of g . The plot of g versus $A_3^{(1)}$ is shown in Fig. 14.

When one real and two complex roots are considered, the equations become

$$g = 4 - 4(Ax) - 2B, \quad (142)$$

$$2g = 3 + 8(Ax)^3 - 6(Ax)A^2 + B^3, \quad (143)$$

$$16g = 5A_3^{(1)} - 64(Ax)^5 - 80(Ax)^3A^2 - 20(Ax)A^4 - 2B^5, \quad (144)$$



- Legend: Curve ① --Two complex roots
 Curve ② --One real and two complex roots, $B=0.5$
 Curve ③ --One real and two complex roots, $B=1.0$
 Curve ④ --Two real and two complex roots,
 $4-4(Ax)=0.5$
 Curve ⑤ --Two real and two complex roots,
 $4-4(Ax)=1.0$
 Curve ⑥ --Two real and two complex roots,
 $4-4(Ax)=1.5$

Fig. 14. Plot of g vs. $A_3^{(1)}$ for Various Root Combinations

where the B term in equation (144) is the value of the added real root. The procedure for solving the equations for $\overset{(1)}{A}_3^\infty$ in terms of g is similar to that for the two complex root case. For each value of the parameter B, a solution for $\overset{(1)}{A}_3^\infty$ equivalent to the two complex root case can be obtained. By proper choices of B in convenient increments between zero and a greatest value, it is possible to take advantage of previous calculations to obtain the variation of $\overset{(1)}{A}_3^\infty$ with g. The plot of g versus $\overset{(1)}{A}_3^\infty$ for the above case is shown in Fig. 14.

The root pairs for four complex roots are expressed as $Ae^{j\theta}$, $Ae^{-j\theta}$, $Ce^{j\psi}$, $Ce^{-j\psi}$. When the Tchebycheff polynomial representation is applied, the result is

$$g = [4 - 4(Ax)] - 4Cy, \quad (145)$$

$$2g = [3 + 8(Ax)^3 - 6(Ax)A^2] + \quad (146)$$

$$8(Cy)^3 - 6(Cy)C^2,$$

$$16g = 5\overset{(1)}{A}_3^\infty + [-64(Ax)^5 + 80(Ax)^3A^2 - \quad (147)$$

$$20(Ax)A^4] - 64(Cy)^5 + 80(Cy)^3C^2 - 20(Cy).$$

The method of obtaining the plot of g versus $\overset{(1)}{A}_3^\infty$ is similar to the method of the two previous cases. The plot of g versus $\overset{(1)}{A}_3^\infty$ for the above case is included in Fig. 14.

For values of $\overset{(1)}{A}_3^\infty$ greater than 6.80 (for the specific example under consideration), introduction of a suitable degenerate zero in N'' reduces $\overset{(1)}{A}_3^\infty$ to 6.80, while retaining the g of two. In the $\overset{(1)}{A}_3^\infty$ range from zero to approximately 5.47, the greatest value of g occurs when the s_{rj}

system consists of two complex roots, which form the simplest consistent combination for solution of the simultaneous equations. In the $A_3^{(1)\infty}$ range from approximately 5.47 to 6.80, no physical solution exists for the s_{rj} system with two complex roots. However, the next simpler case, that of one real and two complex roots, can be used in the $A_3^{(1)\infty}$ range from 5.47 to 6.80 to yield g values almost as large as might have been obtained had the two-root curve been applicable.

While the curves of Fig. 14 are constructed for a particular case, the type of curves and general behavior apply for situations involving a transmission zero of multiplicity two at infinity. The first trial in an optimizing problem for the transmission zero of multiplicity two at infinity should include the two complex zero case where the s_{rj} 's are equal to $Ae^{j\theta}$ and $Ae^{-j\theta}$. Only if the simultaneous equations show a need for θ greater than $\pi/2$, should the next more complex combination be considered. It should be noted that the latter introduces more unknowns than equations. A solution which provides consistency of results must be used. The above approach conceivably can be applied for transmission zeros of complexity greater than two.

Construction of Matching Networks

Tchebycheff behavior.--The basic essentials of the mathematical description of the Tchebycheff behavior and the manner in which the poles and zeros of the reflection and transmission coefficients are related to Tchebycheff functions are given in detail in a previous section. It is appropriate at this point to outline specific methods for obtaining the networks which lead to a predetermined Tchebycheff behavior.

Let it be assumed that the values and locations of the elements of the load and source networks are specified and that the value of the nominal pass band width, ω_c , the total number of zeros of transmission at infinity, n , and an acceptable limit of tolerance and gain are given. The necessary $A^{(1)}_\infty$'s and $A^{(2)}_\infty$'s can be found directly from the load and source networks by means of the methods of Appendix B.

Simultaneous solution of equations of the type specified by equations (106), (107), (109), and (110) leads to values of the parameters a and b and right-half s -plane zero summations. The Tchebycheff poles and zeros of Q_1 can be found by use of equations (100) and (101). The added right-half s -plane zeros of Q_1 introduce pole and zero factors as explained in Chapter II. If maximum gain is a deciding factor, choice of the single right-half s -plane zeros of reflection should be influenced by the material in the section "Optimum Gain Considerations," which is found in this chapter. The double right-half s -plane zeros of reflection are chosen to be consistent with a required tolerance. The combination of the zero and pole factors of Q_1 takes a form similar to equation (74). A value for $Z_1(s)$, that is, the input impedance to the entire network system (including load, matching, and source networks), can be found from an inverted form of equation (6) as

$$Z_1(s) = \frac{1 + \rho(s)}{1 - \rho(s)}. \quad (148)$$

A Darlington synthesis of $Z_1(s)$ leads to the load, matching, and source networks in succession.

Use of degenerate zeros.--If the load network has a zero of transmission of multiplicity n' at infinity, the $A_{2n'-1}^{(1)\infty}$ term is increased algebraically by the presence of a degenerate zero when n' is even and decreased algebraically when n' is odd. A similar statement can be made concerning the source network with a zero of transmission of multiplicity n'' at infinity.

In the manipulation of the simultaneous equations for determining a , b , etc., it may be apparent that a more advantageous solution results if the $A_{2n'-1}^{(1)\infty}$ or $A_{2n''-1}^{(2)\infty}$ terms, or both, can be changed. If the desired changes are consistent with the conditions just stated for degenerate zeros, the matching network can begin or end, or both, in properly chosen elements of the same type as the last elements of the load and/or source networks. For example, let it be assumed that $n'=3.0$, $A_5^{(1)\infty}=4.0$, and the last element of N' (the load network) is a shunt capacitance. If an $A_5^{(1)\infty}$ of three results in more desirable characteristics, the matching network can begin with a shunt capacitance of a value determined by the $A_5^{(1)\infty}$ of three. The remainder of the matching network can be found by absorbing temporarily the degenerate zero element into the load network.

Use of Darlington type C and D sections.--The section which appears as the result of a simple zero of reflection on the positive real axis is called a Darlington type C section (30) and is shown in Fig. 15 in a tee form and in a close-coupled form. It is demonstrated earlier that a zero of transmission occurs at an added right-half s -plane zero of reflection. The shunt M-C combination of Fig. 15 accounts for the right-half s -plane zero of transmission on the real axis. The method

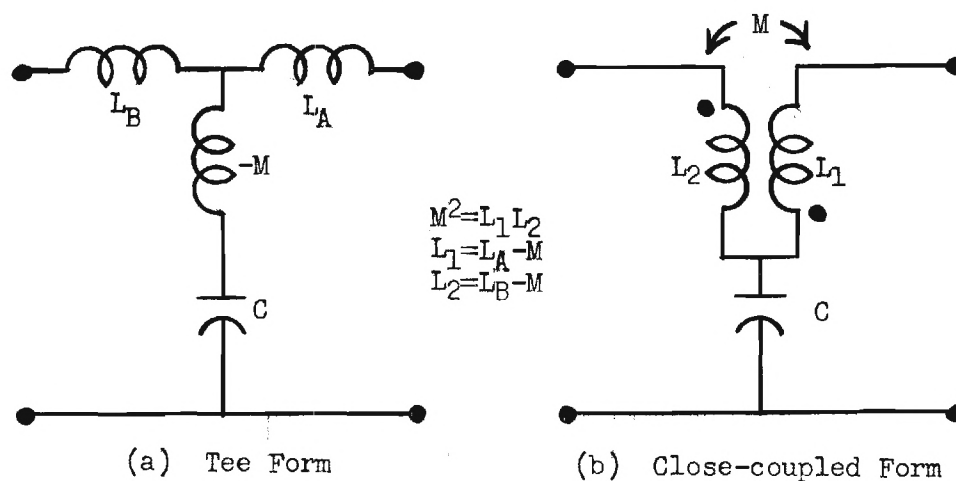


Fig. 15. Darlington Type C Section

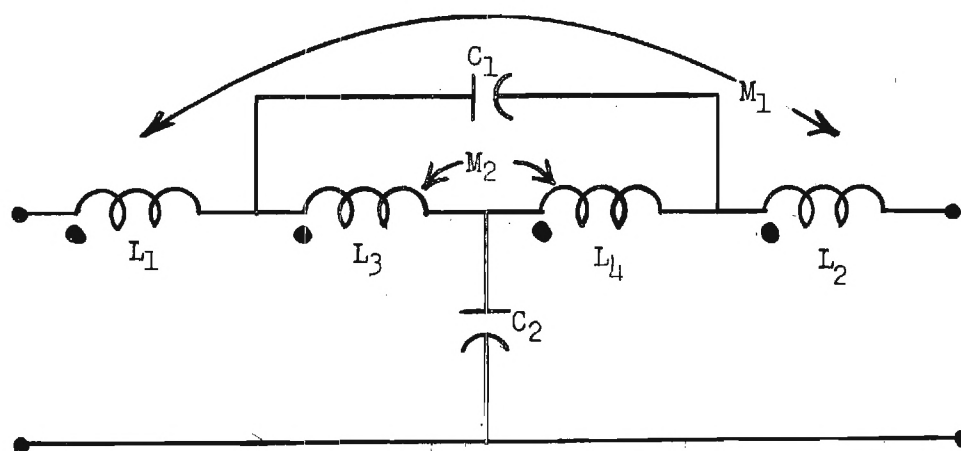


Fig. 16. Darlington Type D Section

of including the type C section in the matching network is shown in Appendix E where examples are worked.

The section which appears as a result of two conjugate right-half s-plane zeros of reflection is called a Darlington type D section and is shown in Fig. 16. Any desired combination of added right-half s-plane

zeros of transmission can be achieved through cascaded type C and D sections.

The addition of a double zero of reflection at a point on the positive real axis introduces a single zero of transmission at the positive real axis point. This fact is verified earlier. Hence, the presence of a double zero of reflection on the positive real axis requires a Darlington type C section in the matching network. An example of the above case is shown in Appendix E.

CHAPTER IV

SUMMARY

This study shows that it is possible to produce an impedance match over a finite frequency range between a source system and a load impedance. The study resolves itself into two major phases: the determination of the conditions imposed on a resistance-terminated cascade of three lossless two-terminal-pair networks by the two outer networks of the cascade, and second, the investigation of the effects of specifying the Tchebycheff form of matching behavior over a finite frequency range.

It is found that the conditions imposed on the overall system by the two outer networks can be expressed as the necessary conditions of physical realizability of the overall reflection coefficients ρ_1 and ρ_2 and the overall transmission coefficient t as determined by the known reflection and transmission coefficients ρ_1' , t' , ρ_2'' , t'' of the load and source networks. The necessary conditions are found to be sufficient as well.

Instead of a reflection or transmission coefficient it is convenient to use the return loss function, which is simply related to its corresponding reflection coefficient, as a means of describing the transmission behavior at real frequencies. A set of integral relationships serves to relate the real frequency behavior of the magnitude of the reflection coefficients to a number of coefficients of the Taylor series expansions for the return losses about transmission zeros.

These coefficients are known through the necessary conditions imposed by the known load and source networks.

With the Tchebycheff (or equal ripple) form of transmission in mind, a second set of integral relationships is developed in terms of a return loss function, the corresponding reflection coefficient function of which has no right-half s -plane zeros. Combination of the two sets of integral relationships and use of suitable additive factors lead to the determination of the overall reflection coefficients. Methods of modern network synthesis are then used to determine the matching network.

The additive factors mentioned above are added right-half s -plane zeros of reflection. They are used to produce consistency in the combined sets of integral equations and are also used to adjust the tolerance or variation between maximum and minimum transmission in the pass band.

For particular load and source networks and a fixed complexity of the overall network, it is possible to (a) obtain any level of transmission less than a fixed limiting value if no restriction is placed on tolerance and (b) obtain any desired tolerance less than a fixed limiting value if no restriction is placed on the level of transmission. (A decrease in tolerance always causes a decrease in the level of transmission.) If both a desired gain and transmission are specified in advance, it may or may not be possible to find a matching network to meet the specified limits. In the event it is not possible to find a suitable solution, a compromise solution must be made.

APPENDIX A

WAVE MATRIX SOLUTION FOR REFLECTION AND TRANSMISSION COEFFICIENTS

Three independent circuit parameters are required to specify an arbitrary two-terminal-pair network completely. These may be either the three open-circuit impedances, z_{11} , z_{22} , and z_{12} , or the three short-circuit admittances, y_{11} , y_{22} , and y_{12} , or one transmission and two reflection coefficients. The situation of interest here is the case of transmission and reflection coefficients of two terminal-pairs.

The expressions for the reflection and transmission coefficients, found below, can be obtained from the wave matrix of two lossless transmission lines terminating in a lossless two terminal-pair (31). The wave matrix is the coefficient matrix of the simultaneous algebraic equations expressing the relationships between incident and reflected voltages and currents at the network junctions. The diagram of Fig. 1-A illustrates the electrical connection of the lines and indicates the notation for the steady-state phasor voltage and current relationships at each end of the two terminal-pair. The phasor voltages at a and b are \dot{V}_a and \dot{V}_b , while the phasor currents are \dot{I}_a and \dot{I}_b . (The dots indicate steady-state phasor forms.) The phasor voltages and phasor currents are further divided into incident and reflected components. The "1" and "3" subscripts refer to the incident wave, and "2" and "4" subscripts refer to the reflected wave. Since the characteristic impedances are pure resistances, the power in the incident and reflected waves is

- $|\dot{A}_1|^2/Z_{oa}$ for the power in the incident wave at a,
 $|\dot{A}_2|^2/Z_{oa}$ for the power in the reflected wave at a,
 $|\dot{A}_3|^2/Z_{ob}$ for the power in the incident wave at b, and
 $|\dot{A}_4|^2/Z_{ob}$ for the power in the reflected wave at b.

The powers are proportional to the squares of the magnitudes of the incident and reflected voltages. If ρ_1 , ρ_2 , and t are designated as the voltage-reflection and voltage-transmission coefficients of the two terminal-pair, the following equations describe the behavior of the voltages and currents at the terminals of the two terminal-pair.

$$\frac{\dot{A}_2}{\sqrt{Z_{oa}}} = \rho_1 \frac{\dot{A}_1}{\sqrt{Z_{oa}}} + t \frac{\dot{A}_4}{\sqrt{Z_{oa}}}, \quad (A-1)$$

$$\frac{\dot{A}_3}{\sqrt{Z_{ob}}} = t \frac{\dot{A}_1}{\sqrt{Z_{ob}}} + \rho_2 \frac{\dot{A}_4}{\sqrt{Z_{ob}}}. \quad (A-2)$$

The coefficient matrix of the above system is the wave matrix.

If the characteristic impedance of each transmission line is one ohm, the lines can be replaced by one-ohm resistances and associated sources as shown in Fig. 2-A. The same directions are used for voltage and current as before, but the dots are no longer used in the phasor representation. The incident voltage on the right is $E_1/2$ and is $E_2/2$ on the left. Since the total voltage at the right-hand network terminals is V_1 , the reflected voltage is $(V_1 - E_1/2)$. Therefore, the following equations are the counterparts of equations (A-1) and (A-2):

$$\left(V_1 - \frac{E_1}{2}\right) = \rho_1 \frac{E_1}{2} + t \frac{E_2}{2}, \quad (A-3)$$

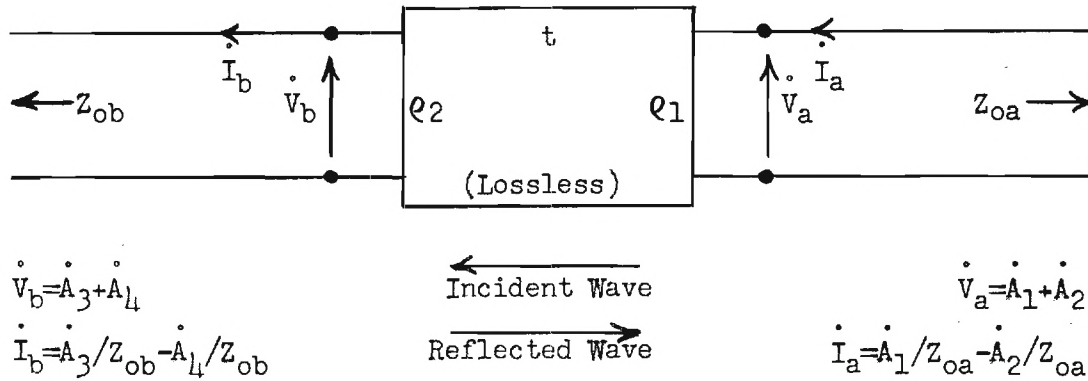


Fig. 1-A. Lossless Lines Connected to Two Terminal-pair

$$\left(V_2 - \frac{E_2}{2}\right) = t \frac{E_1}{2} + \rho_2 \frac{E_2}{2}. \quad (\text{A-4})$$

From equations (A-3) and (A-4) and the definitions of the reflection and transmission coefficients, it is evident that

$$\rho_1 = \frac{2V_1 - E_1}{E_1} \bigg|_{E_2=0}, \quad (\text{A-5})$$

$$\rho_2 = \frac{2V_2 - E_2}{E_2} \bigg|_{E_1=0}, \quad (\text{A-6})$$

$$t = \frac{2V_2}{E_1} \bigg|_{E_2=0} = \frac{2V_1}{E_2} \bigg|_{E_1=0}. \quad (\text{A-7})$$

It is seen from Fig. 2-A and equations (A-5) and (A-6) that ρ_1 and ρ_2 are the conventional voltage-reflection coefficients between impedances Z_1 and Z_2 , respectively, and one-ohm terminations. It can be shown also that (32)

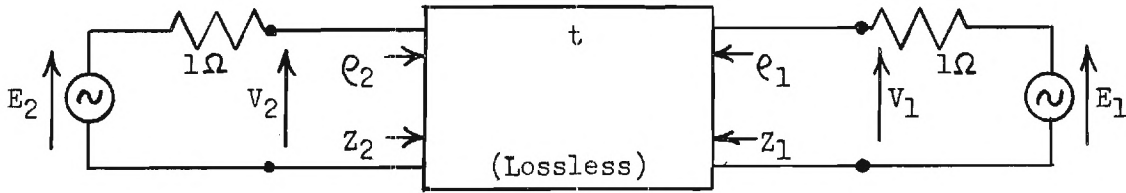


Fig. 2-A. One-ohm Terminated Network

$$e_1 = \frac{Z_1 - 1}{Z_1 + 1}, \quad (\text{A-8})$$

$$e_2 = \frac{Z_2 - 1}{Z_2 + 1}. \quad (\text{A-9})$$

In this study it is necessary to examine the reflection and transmission coefficients of two and three lossless sections in tandem. For this purpose, the so-called **A**-matrix given in Ragan (33) is introduced. The elements of this matrix are the **A**-coefficients. The **A**-coefficients are alternate means of describing two-terminal-pair network behavior in terms of the input and output voltages. The matrix equation for the terminal voltages in terms of the **A**-coefficients is

$$\begin{vmatrix} \frac{E_1}{2} \\ V_1 - \frac{E_1}{2} \end{vmatrix} = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \times \begin{vmatrix} \frac{E_2}{2} \\ V_2 - \frac{E_2}{2} \end{vmatrix}. \quad (\text{A-10})$$

Through use of equations (A-3), (A-4), and (A-10), it is found that

$$A_{11} = 1/t, \quad (\text{A-11})$$

$$A_{12} = -\rho_2/t, \quad (A-12)$$

$$A_{21} = \rho_1/t, \quad (A-13)$$

$$A_{22} = t - \rho_1 \rho_2 / t. \quad (A-14)$$

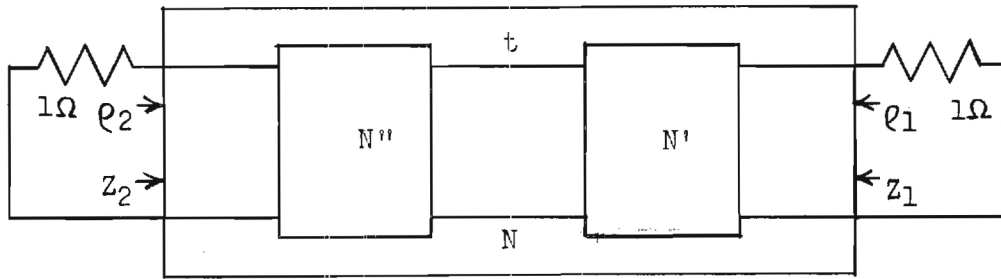
It can be shown that the A-matrix of any number of cascaded lossless two terminal-pairs is the product of the individual A-matrices (34). This useful property permits calculation of the overall reflection and transmission coefficients of the duo-cascade of two lossless two terminal-pairs in terms of the transmission and reflection coefficients of the individual two terminal-pairs. The arrangement of the duo-cascade is shown in Fig. 3-A. Through use of the product of the A-matrices of the individual two terminal-pairs, the reflection and transmission coefficients for the duo-cascade are found to be

$$\rho_1 = \rho_1' + \rho_1'' \frac{(t')^2}{1 - \rho_2' \rho_1''}, \quad (A-15)$$

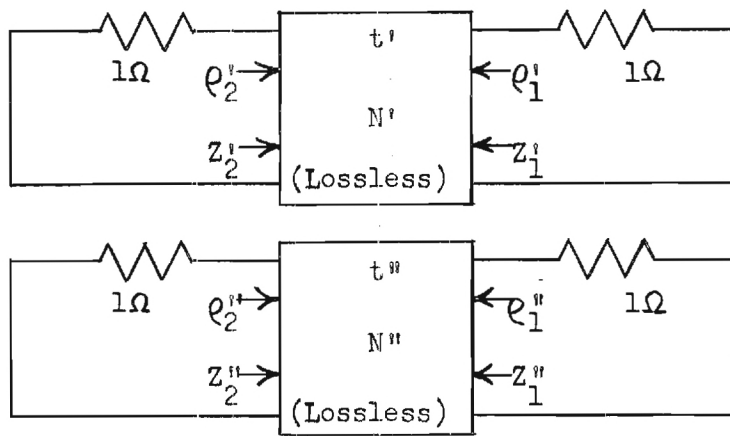
$$\rho_2 = \rho_2'' + \rho_2' \frac{(t'')^2}{1 - \rho_2' \rho_1''}, \quad (A-16)$$

$$t = \frac{t' t''}{1 - \rho_2' \rho_1''}. \quad (A-17)$$

In the same way the reflection and transmission coefficients for the trio-cascade are shown in Fig. 4-A. The appropriate equations follow as



(a) Duo-cascade



(b) Individual Networks

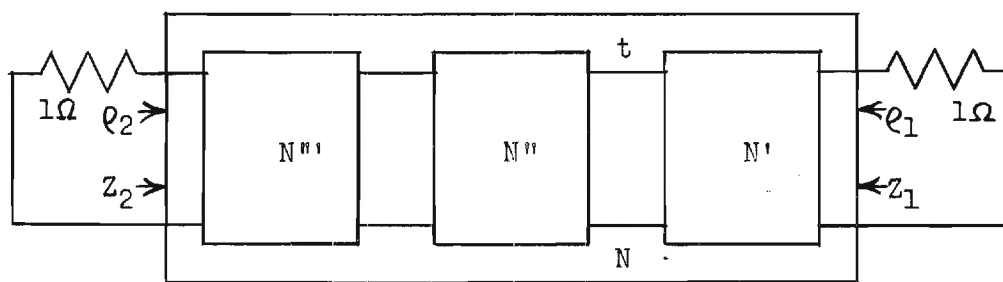
Fig. 3-A. Duo-cascade of Two Terminal-pairs and Individual Networks

$$e_1 = e_1' + [e_1''(1 - e_2''e_1''') + (t'')^2 e_1''] \cdot \quad (\text{A-18})$$

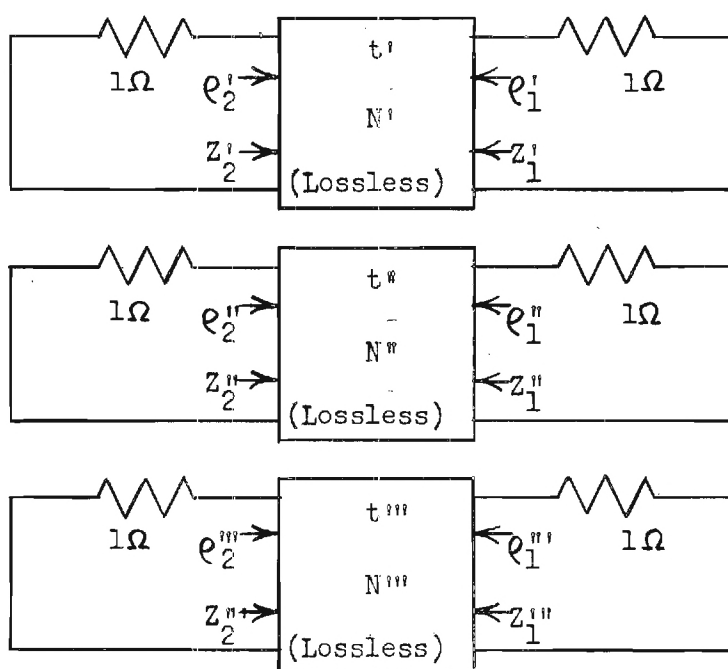
$$\left[\frac{(t')^2}{(1 - e_2' e_1'')(1 - e_2'' e_1''') - e_2' e_1'' (t'')^2} \right],$$

$$e_2 = e_2''' + [e_2''(1 - e_1''e_2') + (t'')^2 e_2'] \cdot \quad (\text{A-19})$$

$$\left[\frac{(t'')^2}{(1 - e_2' e_1'')(1 - e_2'' e_1''') - e_2' e_1'' (t'')^2} \right],$$



(a) Trio-cascade



(b) Individual Networks

Fig. 4-A. Trio-cascade of Two Terminal-pairs and Individual Networks

$$t = \left[\frac{t' t'' t'''}{(1 - e_2' e_1'')(1 - e_2'' e_1''') - e_2' e_1'' (t'')^2} \right]. \quad (\text{A-20})$$

APPENDIX B

DETERMINATION OF COEFFICIENTS FOR RETURN LOSS EXPANSIONS

General Properties of Return Loss Expansions

In this appendix, ρ and t are general reflection and transmission coefficients for any lossless two terminal-pair. Correspondingly, $\ln(1/\rho(s))$ is a general return loss function. The coefficients of the Taylor series expansion of $\ln(1/\rho)$ about a zero of transmission are proportional to the derivatives of $\ln(1/\rho)$ at the zero of transmission. It can be shown (35) that the Taylor series expansions of $\ln(1/\rho(s))$ about zeros of t of order n at the origin and infinity can have only odd numbered and real coefficients up to and including the $(2n-1)$ th. When the zero of transmission is at a point on the j -axis other than $s=0$ and $s=\infty$, all coefficients of the Taylor expansion of $\ln(1/\rho(s))$ up to and including the $(2n-1)$ th can be present. The odd coefficients are real and the even coefficients are imaginary.

Derivatives of Return Loss Expansions

A reflection coefficient can be expressed as

$$\rho(s) = \pm h \frac{(s-s_{o1})(s-s_{o2})\dots(s-s_{om})}{(s-s_{p1})(s-s_{p2})\dots(s-s_{pm})}, \quad (B-1)$$

where $s_{o1}, s_{o2}, \dots, s_{om}$ are the zeros of $\rho(s)$, $s_{p1}, s_{p2}, \dots, s_{pm}$ are the poles of $\rho(s)$, and h is a real constant. A corresponding expression for $\ln(1/\rho(s))$ is

$$\ln \frac{1}{\rho(s)} = \ln \frac{1}{\pm h} + \left[\ln(s-s_{p1}) + \dots + \ln(s-s_{pm}) \right] - \left[\ln(s-s_{o1}) + \dots + \ln(s-s_{om}) \right]. \quad (B-2)$$

The first derivative of $\ln(1/\rho(s))$ yields

$$\left[\frac{1}{(s-s_{p1})} + \frac{1}{(s-s_{p2})} + \dots + \frac{1}{(s-s_{pm})} \right] - \left[\frac{1}{(s-s_{o1})} + \frac{1}{(s-s_{o2})} + \dots + \frac{1}{(s-s_{om})} \right]. \quad (B-3)$$

The second derivative of $\ln(1/\rho(s))$ yields

$$-\left[\frac{1}{(s-s_{p1})^2} + \frac{1}{(s-s_{p2})^2} + \dots + \frac{1}{(s-s_{pm})^2} \right] + \left[\frac{1}{(s-s_{o1})^2} + \frac{1}{(s-s_{o2})^2} + \dots + \frac{1}{(s-s_{om})^2} \right]. \quad (B-4)$$

The kth derivative of $\ln(1/\rho(s))$ becomes

$$(k-1)! \left\{ \left[\frac{1}{(s_{o1}-s)^k} + \dots + \frac{1}{(s_{om}-s)^k} \right] - \left[\frac{1}{(s_{p1}-s)^k} + \dots + \frac{1}{(s_{pm}-s)^k} \right] \right\}. \quad (B-5)$$

Return Loss Coefficients at Transmission Zeros

Return loss coefficients for zeros of t at the origin.--From the previous discussion of return loss expansions about transmission zeros at $s=0$, $s=\infty$, and $s=j\omega$, it is evident that the Taylor series expansion of $\ln(1/\rho(s))$ about a zero of transmission of multiplicity n at the origin can be expressed in the form

$$\ln 1/\rho(s) = j\beta + A_1^0 s + A_3^0 s^3 + \dots + A_{2n-3}^{2n-3} s^{2n-3} + A_{2n-1}^{2n-1} s^{2n-1} + \dots, \quad (B-6)$$

where the A 's are real. The real part of $\ln(1/\rho(s))$ vanishes at the origin for a zero of t at the origin, leaving zero or π as the value of β , depending on the sign of ρ at the origin. By letting $s=0$ in

expression (B-5), the A^0 's can be found directly as

$$A_{2k-1}^0 = \frac{1}{2k-1} \left(\sum_i s_{oi}^{-(2k-1)} - \sum_i s_{pi}^{-(2k-1)} \right), \quad (B-7)$$

where $k=1, 2, 3, \dots, n$, and where the s_{oi} 's and the s_{pi} 's are the zeros and poles, respectively, of ρ .

Return loss coefficients for zeros of t at infinity.--If s is replaced by $1/p$, the behavior of the derivative of $\ln(1/\rho)$ at $s=\infty$ can be examined by noting the behavior of the derivative of $\ln(1/\rho)$ at $p=0$. The general form for the k th derivative becomes

$$(k-1)! \left\{ \left[\frac{1}{\left(\frac{1}{s_{oi}} - p\right)^k} + \frac{1}{\left(\frac{1}{s_{om}} - p\right)^k} \right] - \left[\frac{1}{\left(\frac{1}{s_{pi}} - p\right)^k} + \frac{1}{\left(\frac{1}{s_{pm}} - p\right)^k} \right] \right\}. \quad (B-8)$$

According to the previous discussion of return loss coefficients, the Taylor series expansion of $\ln(1/\rho(s))$ about a zero of transmission at infinity can be expressed as

$$\ln \frac{1}{\rho(s)} = j\beta + A_1^\infty \frac{1}{s} + A_3^\infty \frac{1}{s^3} + \dots + A_{2n-3}^\infty \frac{1}{s^{2n-3}} + A_{2n-1}^\infty \frac{1}{s^{2n-1}} + \dots \quad (B-9)$$

The value of β depends on the sign of ρ at infinity. The A^∞ 's can be found from expression (B-8) by letting $p=0$. The values for the A^∞ 's are, specifically,

$$A_{2k-1}^\infty = \frac{1}{2k-1} \left(\sum_i s_{oi}^{2k-1} - \sum_i s_{pi}^{2k-1} \right), \quad (B-10)$$

where $k=1, 2, 3, \dots, n$.

Return loss coefficients for zeros of t at s equal to $j\omega_v$.--The Taylor series expansion for $\ln(1/\rho(s))$ about a zero of transmission at $s=j\omega_v$ is given by

$$\ln 1/\rho(s) = jB_0^{\omega_v} + A_1^{\omega_v}(s-j\omega_v) + jB_2^{\omega_v}(s-j\omega_v)^2 + \quad (B-11)$$

$$A_3^{\omega_v}(s-j\omega_v)^3 + \dots + jB_{2n-2}^{\omega_v}(s-j\omega_v)^{2n-2} + A_{2n-1}^{\omega_v}(s-j\omega_v)^{2n-1} + \dots$$

The form of equation (B-11), i.e., odd coefficients real, even coefficients imaginary, was stated earlier. The coefficients of equation (B-11) are given as

$$jB_0^{\omega_v} = [\ln 1/\rho(s)]_{s=j\omega_v}, \quad (B-12)$$

$$jB_k^{\omega_v}, A_k^{\omega_v} = \frac{1}{k} \left[\sum_i (s_{oi} - j\omega_v)^k - \sum_i (s_{pi} - j\omega_v)^k \right], \quad (B-13)$$

where $k=1, 2, 3, \dots, 2n-1$. Equation (B-13) follows directly from expression (B-5). The odd values of k apply for the $A_k^{\omega_v}$'s, and the even values of k apply for the $B_k^{\omega_v}$'s.

Return loss coefficients for zeros of t on the positive real axis.--

When t has a zero of multiplicity n at $s=\sigma_v$, a point on the positive real axis, it is possible for ρ to have a zero of multiplicity n_0 there also. The manner of finding the Taylor series of $\ln(1/\rho(s))$ with both n_0 and n indices is outlined in the following sections.

It is stated in the literature (36) that ρ has a pole of multiplicity n_0 at the point $-\sigma_v$ whenever ρ has a zero of multiplicity n_0 at σ_v .

If ρ has a zero of multiplicity n_0 at σ_v , the function $\ln(1/\rho)$ is

not analytic at σ_v and cannot be expanded in a Taylor series about σ_v . However, $\ln(1/\rho)$ can be expressed in terms of a modified return loss which is analytic at σ_v . In order to secure a modified reflection coefficient which does not have zeros in the right-half s-plane, $\rho(s)$ can be multiplied by $[(s+\sigma_v)/(s-\sigma_v)]^{n_0}$. The modified reflection coefficient is given as

$$\rho^{(n_0)}(s) = \rho(s) \frac{(s+\sigma_v)^{n_0}}{(s-\sigma_v)^{n_0}}, \quad (\text{B-14})$$

where $\rho^{(n_0)}(s)$ does not have zeros at $s=\sigma_v$. The corresponding expression for $\rho(s)$ is

$$\rho(s) = \rho^{(n_0)}(s) \frac{(s-\sigma_v)^{n_0}}{(s+\sigma_v)^{n_0}}. \quad (\text{B-15})$$

The expanded form for $\ln(1/\rho(s))$ about a zero of t on the positive real axis becomes

$$\begin{aligned} \ln 1/\rho(s) = & n_0 \ln[(s+\sigma_v)/(s-\sigma_v)] + A_0^{\sigma_v} + \quad (\text{B-16}) \\ & j\beta + A_2^{\sigma_v}(s-\sigma_v)^2 + \dots + \\ & A_{2n-n_0-2}^{\sigma_v}(s-\sigma_v)^{2n-n_0-2} + A_{2n-n_0-1}^{\sigma_v}(s-\sigma_v)^{2n-n_0-1} + \dots \end{aligned}$$

The coefficients of equation (B-16) are real since $\rho(s)$ is real for real s . It is seen that

$$A_0^{\sigma_v} + j\beta = \ln \left\{ \rho(s) \left[\frac{(s+\sigma_v)}{(s-\sigma_v)} \right]^{n_0} \right\}_{s=\sigma_v}, \quad (\text{B-17})$$

$$A_k^{\sigma_v} = \frac{1}{k} \left[\sum_i (s_{oi} - \sigma_v)^{-k} - \sum_i (s_{pi} - \sigma_v)^{-k} \right], \quad (B-18)$$

where $k=1, 2, 3, \dots (2n-n_0-1)$. The zero at σ_v and the pole at $-\sigma_v$ should not be included in the summations of equation (B-18) because the $A_k^{\sigma_v}$ coefficients as determined by equation (B-18) are, in reality, the $A_k^{\sigma_v}$ coefficients for $\ln(1/\rho(s))$; $\rho(s)$ does not contain zeros at $+\sigma_v$ or poles at $-\sigma_v$.

Return loss coefficients for zeros of t in the right-half s -plane.--In a similar manner, the appropriate equations for a zero of t of multiplicity n at a point s_v in the right-half s -plane are found. The equation for $\ln(1/\rho(s))$ is

$$\begin{aligned} \ln 1/\rho(s) = & n_0 \ln \left[\frac{(s+s_v)(s+\bar{s}_v)}{(s-s_v)(s-\bar{s}_v)} \right] + \quad (B-19) \\ & (A_1^{s_v} + jB_1^{s_v})(s-s_v) + (A_2^{s_v} + jB_2^{s_v})(s-s_v)^2 + \\ & \dots (A_{2n-n_0-2}^{s_v} + jB_{2n-n_0-2}^{s_v})(s-s_v)^{2n-n_0-2} + \\ & (A_{2n-n_0-1}^{s_v} + jB_{2n-n_0-1}^{s_v})(s-s_v)^{2n-n_0-1} + \dots \end{aligned}$$

The coefficients of equation (B-19) are given by

$$A_0^{s_v} + jB_0^{s_v} = -\ln \left[\rho(s) \left(\frac{(s+s_v)(s+\bar{s}_v)}{(s-s_v)(s-\bar{s}_v)} \right)^{n_0} \right], \quad (B-20)$$

$$A_k^{s_v} + jB_k^{s_v} = \frac{1}{k} \left[\sum_i (s_{oi} - s_v)^{-k} - \sum_i (s_{pi} - s_v)^{-k} \right], \quad (B-21)$$

where $k=1, 2, 3, \dots (2n-n_0-1)$. The zeros at s_v and \bar{s}_v and the poles at $-s_v$ and $-\bar{s}_v$ should not be included in the summations of equation

(B-21) because the $(A_k^{s_v} + jB_k^{s_v})$ coefficients as determined by equation (B-21) are the coefficients for $\ln(1/Q(s))$; $Q(s)$ does not contain zeros at s_v and \bar{s}_v or poles at $-s_v$ and $-\bar{s}_v$. The expansion for $\ln(1/Q(s))$ at a zero of t equal to \bar{s}_v is the conjugate of equation (B-19).

Methods for Calculating Coefficients of Taylor Series of $\ln(1/Q)$

If the reflection coefficient is given in factored form, as in equation (B-1), the direct application of the formulas just developed is sufficient to determine the Taylor series coefficients. Unfortunately, however, the reflection coefficients are found very often in the form

$$Q(s) = \frac{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s + a_n}{b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} + \dots + b_{n-1} s + b_n}. \quad (B-22)$$

Two methods for calculating the coefficients of the Taylor series of $\ln(1/Q(s))$ about a transmission zero are apparent immediately. First, the numerator and denominator of equation (B-22) can be factored and the various s_{oi} 's and s_{pi} 's found; second, the successive derivatives of $\ln(1/Q(s))$ can be calculated and evaluated at the transmission zero. Both of these methods are extremely cumbersome. When the degree of the numerator or denominator of $Q(s)$ is greater than four, the first method requires the use of one of the approximate root determination schemes, whereas the labor for the second method pyramids with each successive derivative.

The form of the expressions for the coefficients given by equations (B-7), (B-10), (B-13), (B-18), and (B-20) suggests that direct summations of the roots can be used. By a method due to Newton

(37), the sum of the m th powers of the roots of an algebraic equation with real coefficients can be found. For example, if a function of s (such as the numerator of $\rho(s)$) is given by

$$f(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s + a_n = 0, \quad (\text{B-23})$$

the sum of the m th powers of the roots assumes the form

$$S_m = S_{o1}^m + S_{o2}^m + \dots + S_{on}^m = \sum_i S_{oi}^m. \quad (\text{B-24})$$

The successive values of S_m are found from the following tabulation starting with S_1 :

$$1a_1 + S_1 a_0 = 0 \quad (\text{B-25})$$

$$2a_2 + S_1 a_1 + S_2 a_0 = 0$$

$$3a_3 + S_1 a_2 + S_2 a_1 + S_3 a_0 = 0$$

$$\begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} .$$

Tabulation (B-25) is used to find the coefficients for the Taylor series expansion of $\ln(1/\rho(s))$ at a zero of transmission at infinity where the summations appear in the form

$$A_{2k-1}^\infty = \frac{1}{2k-1} \left(\sum_i S_{oi}^{2k-1} - \sum_i S_{pi}^{2k-1} \right). \quad (\text{B-26})$$

The $\sum_i S_{oi}^{2k-1}$ and $\sum_i S_{pi}^{2k-1}$ are the numerator and denominator S_{2k-1} 's, respectively.

The sums of the m th powers of the reciprocals of the roots of equation (B-23) can be found by repeating tabulation (B-25) for a

transformed equation, the roots of which are the reciprocals of the roots of equation (B-23). The result is

$$1a_{n-1} + S_1a_n = 0 \quad (B-27)$$

$$2a_{n-2} + S_1a_{n-1} + S_2a_n = 0$$

$$3a_{n-3} + S_1a_{n-2} + S_2a_{n-1} + S_3a_n = 0$$

$$\begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} .$$

The tabulation of (B-27) is used to find the coefficients for the Taylor series expansion of $\ln(1/\rho(s))$ about a zero of transmission at the origin where the summations appear in the form

$$A_{2k-1}^0 = \frac{1}{2k-1} \left(\sum_i S_{oi}^{-(2k-1)} - \sum_i S_{pi}^{-(2k-1)} \right). \quad (B-28)$$

The $\xi_{oi}^{-(2k-1)}$ and $\xi_{pi}^{-(2k-1)}$ are the numerator and denominator S_{2k-1} 's, respectively.

The sums of the reciprocals of the diminished roots of equation (B-23) can be found by transforming equation (B-23) into a new equation, the roots of which are s less than the roots of equation (B-23), and then repeating the tabulation of (B-27) with proper regard for the new coefficient values. The above procedure can be used to find the summations required for zeros of transmission on the j -axis and in the right-half s -plane.

APPENDIX C

EFFECT OF DEGENERATE ZEROS

General

When two lossless networks, N' and N'' , are connected in tandem (see Fig. 1-C), there are certain properties of the combined system which are independent of the parameters of N'' and certain properties which are independent of the parameters of N' . For example, at a zero of t' in the right-half s -plane or on the imaginary axis, the reflection coefficient ρ_1 and certain derivatives of ρ_1 are independent of network N'' . At a zero of t'' , a similar statement applies for ρ_2 and network N' .

The expression for the reflection coefficient ρ_1 is repeated from equation (A-15) as

$$\rho_1 = \rho_1' + \rho_1'' \frac{(t')^2}{1 - \rho_2' \rho_1''} = \rho_1' \left(1 + \frac{\rho_1''}{\rho_1'} \frac{(t')^2}{1 - \rho_2' \rho_1''} \right). \quad (C-1)$$

It should be noted that the reflection and transmission coefficients are those for lossless networks with one-ohm terminations. At a zero of t' of multiplicity n' , equation (C-1) indicates that the value of ρ_1 and its first $(2n'-1)$ derivatives are equal to ρ_1' and its first $(2n'-1)$ derivatives at the zero of t' except when $(1 - \rho_2' \rho_1'')$ also has a zero at the zero of t' . Definition: A zero of $(1 - \rho_2' \rho_1'')$ at a zero of t' is called a degenerate zero.

The function $f(s) = (t')^2 / (1 - \rho_2' \rho_1'')$ is undefined at the degenerate

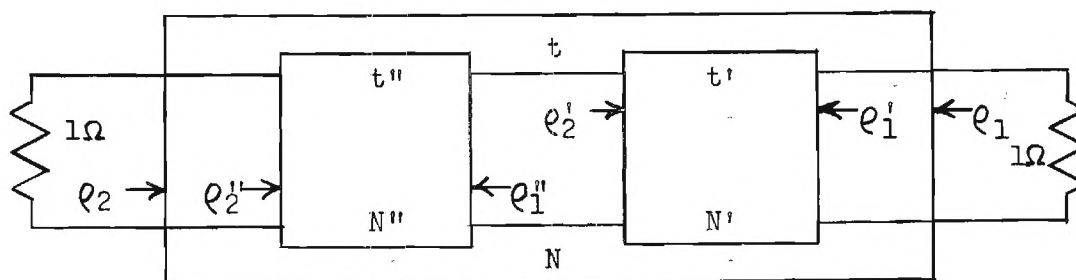


Fig. 1-C. Location of Network Parameters

zero. However, the function $(1 - \rho_2' \rho_1'')$ has at most a simple zero, as proved below in Theorem 1-C, and the function $(t')^2$ has at least a simple zero. The singularity at a degenerate zero is removable by defining $f(s)$ to be zero at a degenerate zero. Moreover, by Theorem 1, Chapter III, the multiplicity of the zero of $(t')^2 / (1 - \rho_2' \rho_1'')$ at a degenerate zero is $(2n' - 1)$. In turn, the number of equalities between the derivatives of ρ_1 and ρ_1' and derivatives of $\ln(1/\rho_1(s))$ and $\ln(1/\rho_1'(s))$ is $(2n' - 2)$.

Theorem 1-C: The degenerate zeros are simple, they occur on the imaginary axis, and they are caused by the first zero-producing elements in N'' .

Proof: The reflection coefficient of a lossless network terminated in a one-ohm resistance is given in terms of the input impedance as

$$\rho_1 = \frac{Z_i - 1}{Z_i + 1} \quad (C-2)$$

Specifically, for Fig. 1-C the reflection coefficients are

$$\rho_1' = \frac{Z_1' - 1}{Z_1' + 1}, \quad (C-3)$$

$$\rho_1'' = \frac{Z_1'' - 1}{Z_1'' + 1}, \quad (C-4)$$

$$\rho_2' = \frac{Z_2' - 1}{Z_2' + 1}. \quad (C-5)$$

With the aid of equations (C-4) and (C-5), the expression $(1 - \rho_2' \rho_1'')$ can be found in terms of individual input impedances as

$$(1 - \rho_2' \rho_1'') = \frac{2Z_2' + 2Z_1''}{(Z_2' + 1)(Z_1'' + 1)}. \quad (C-6)$$

It is shown in the proof of Theorem 1, Chapter III, that $(1 - \rho_2' \rho_1'')$ of equation (C-6) is positive real and has neither poles nor zeros in the right-half s-plane but may have simple zeros or poles on the imaginary axis. However, the zeros or poles on the imaginary axis must be simple because of the positive-real property. Thus, a degenerate zero can occur for s values on the imaginary axis but not in the right-half s-plane. In addition, a degenerate zero can reduce the order of the equal derivatives of $\ln(1/\rho_1(s))$ and $\ln(1/\rho_1'(s))$ at the zero of t' from $(2n'-1)$ to $(2n'-2)$ and thereby affect only the $(2n'-1)$ th coefficient of the Taylor series expansion for $\ln(1/\rho_1(s))$. If the Darlington representation (§8) of N' and N'' , i.e., a ladder configuration of alternate shunt and series zero-producing elements, is assumed, the value of the first zero-producing element of N' determines the A_1 coefficient of the Taylor series expansion of $\ln(1/\rho_1)$ at a zero of t' ,

the first and second zero-producing elements together determine the next coefficient, etc. Thus, the last element of N' has no effect on the $(2n'-2)$ th or preceding coefficients, but does partially determine the $(2n'-1)$ th coefficient. Since the $(2n'-1)$ th coefficient is not independent of N'' in the case of a degenerate zero, the characteristics of N'' partially determine the $(2n'-1)$ th coefficient. If the identity of N'' is abandoned temporarily and the first zero-producing element of N'' is absorbed into N' , the new $(2n'-1)$ th coefficient is independent of the modified N'' , indicating that a degenerate zero is caused by a combination of the last zero-producing element of N' and the first zero-producing element of N'' .

At a zero of $(1-\rho_2'\rho_1'')$ on the imaginary axis, the steady-state phasor representations of ρ_2' and ρ_1'' are

$$\rho_2' = 1 \angle \theta, \quad (C-7)$$

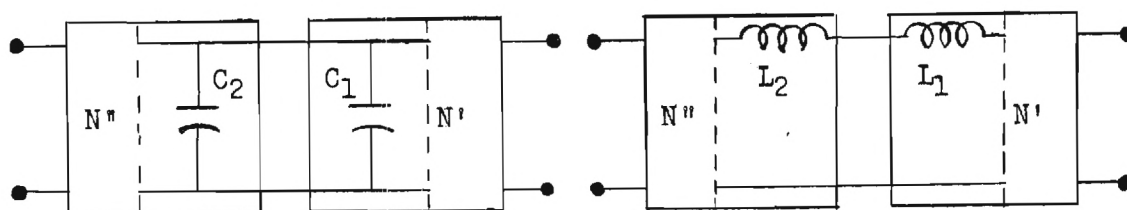
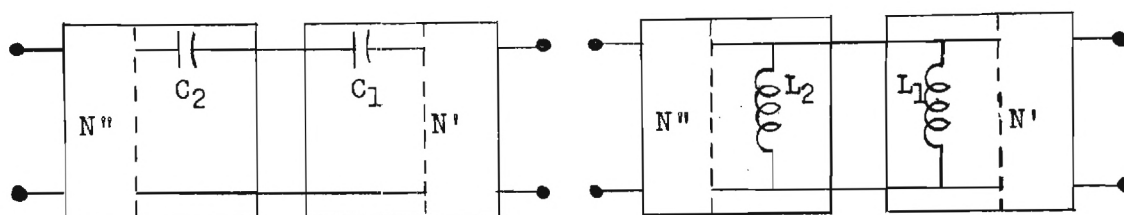
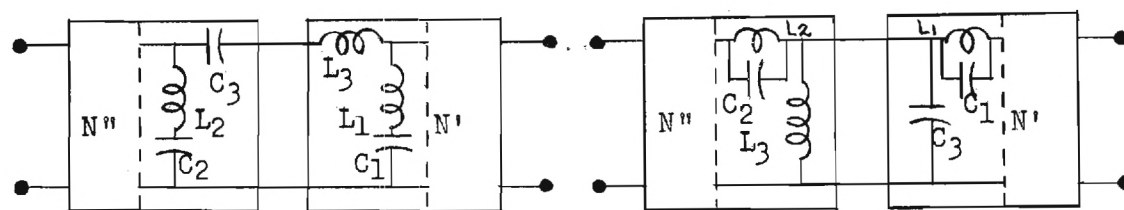
$$\rho_1'' = 1 \angle -\theta. \quad (C-8)$$

Through use of equations (C-5) and (C-4), it is seen that

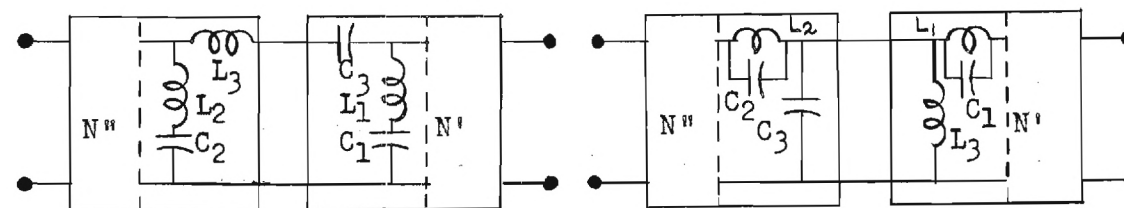
$$Z_2' = \frac{2j \sin \theta}{2 - 2 \cos \theta}, \quad (C-9)$$

$$Z_1'' = \frac{-2j \sin \theta}{2 - 2 \cos \theta}. \quad (C-10)$$

at a j -axis value of s for which a degenerate zero occurs. Under these circumstances, the sum of the impedances Z_2' and Z_1'' is zero. It should be remembered that a zero of $(1-\rho_2'\rho_1'')$ cannot occur unless t' also has a

(a) Degenerate Zeros at $s = \infty$ (b) Degenerate Zeros at $s = 0$ 

$$\omega_v^2 = 1/L_1 C_1 = 1/L_2 C_2 = 1/L_3 C_3$$



$$\omega_v^2 = 1/L_1 C_1 = 1/L_2 C_2 = 1/L_3 C_3$$

(c) Degenerate Zeros at $s = j\omega_v$

Fig. 2-C. Degenerate Zeros of Transmission

zero. Fig. 2-C illustrates arrangements of circuit elements which produce degenerate zeros.

It is apparent that the presence of a degenerate zero affects the $(2n'-1)$ th coefficient of the Taylor series expansion for $\ln(1/\rho_1)$ about a zero of t' of multiplicity n' . It can be shown (39) that for n' odd (or even) the $(2n'-1)$ th term of the Taylor expansion of $\ln(1/\rho_1)$ about a zero of t' is always decreased (or increased) algebraically in the presence of a degenerate zero. Likewise, for n''' odd (or even) the $(2n'''-1)$ th term of the Taylor expansion of $\ln(1/\rho_2)$ about a zero of t''' is always decreased (or increased) algebraically in the presence of a degenerate zero.

APPENDIX D

DEVELOPMENT OF CONTOUR INTEGRALS

Contour Integrals Involving ρ_0

The return loss expressions $\ln(1/\rho_1)$ and $\ln(1/\rho_2)$ for the complete system can be expanded in Taylor series about transmission zeros in the s -plane at zero, infinity, and on the j -axis, since the return loss expressions are analytic at the above points. In the right-half s -plane, however, it is possible for both a reflection and transmission zero to occur at the same point (40). In this case, the return loss expressions become indeterminate. The difficulty is resolved by expanding a modified return loss about a transmission zero. The modified function is analytic at the transmission zero in the right-half s -plane. This behavior is described in Appendix B.

Relationships between the coefficients of the Taylor series of the return loss expressions and the behavior of $\ln(1/|\rho_1|)$ and/or $\ln(1/|\rho_2|)$ at real frequencies can be derived, if desired, by application of Cauchy's integral formula (41) to the Taylor series expansions of the return losses. The contour used here for application of Cauchy's integral formula consists of a semicircle in the right-half s -plane, center at the origin, radius R , and that portion of the imaginary axis between $-jR$ and $+jR$. In the usual way, the radius of the semicircle is allowed to become infinite. Appropriate indentations are made on the path along the imaginary axis to avoid singularities in the integrand. The first step in evaluating a particular

coefficient of a Taylor series expansion would be to form the product of an appropriate weighting function with the return loss. The product function contains poles of appropriate multiplicity so that integration of the product function around the contour and passage to the limit as indicated above results in a number from which the desired coefficient can be obtained.

In the event that ρ_1 has zeros in the right-half s-plane, it is desirable, for the purposes of this study, to obtain a new reflection coefficient function ρ_0 . The new function has the poles of ρ_1 and has the left-half s-plane zeros of ρ_1 plus the shifted right-half s-plane zeros of ρ_1 . The latter zeros are found by shifting the right-half s-plane zeros of ρ_1 to diametrically opposite locations in the left-half s-plane. The result is a reflection coefficient function ρ_0 which has no right-half s-plane zeros but has the magnitude of ρ_1 at real frequencies. In this regard, it is shown in equation (7) that ρ_2 has the same magnitude as ρ_1 at real frequencies.

The new return loss function is given as

$$\ln \frac{1}{\rho_0} = \ln \left[\pm \frac{1}{h} \frac{\prod_{i=1}^n (s - s_{pi})}{\prod_{k=1}^n (s + s_{ok})} \right]. \quad (D-1)$$

The zeros of ρ_0 are the s_{ok} 's which lie in the left-half s-plane. The poles of ρ_0 lie in the left-half s-plane, since it was shown in Chapter II that a realizable reflection coefficient function is analytic in the right-half s-plane and on the j-axis. The sign is chosen so that ρ_0 is positive at the origin. Thus, ρ_0 and $\ln(1/\rho_0)$ are analytic at transmission zeros in the right-half s-plane and on

the j -axis and can be expanded in Taylor series about those transmission zeros. Each of the coefficients of the Taylor series for $\ln(1/\varrho_0)$ at a transmission zero can be expressed in terms of the zeros and poles of ϱ_0 , using the methods of Appendix B.

The evaluation of the coefficients then can be made in terms of the behavior of $\ln(1/|\varrho_0|)$ along the imaginary axis, thus establishing a link between the real frequency behavior of $\ln(1/|\varrho_0|)$ and the zeros and poles of ϱ_0 . The procedure for evaluating the coefficients of the Taylor expansion for $\ln(1/\varrho_0)$ about a transmission zero involves application of Cauchy's integral formula to $\ln(1/\varrho_0)$ and use of the integration contour previously described. It is worthy of note that $\ln(1/\varrho_0)$ has no singularities within the integration contour. It can be shown (42) that along the j -axis $\varrho_0(s) = |\varrho_0(s)|e^{j\theta(s)}$, $|\varrho_0(s)|$ and $\ln(1/|\varrho_0|)$ are even functions of s , and θ is an odd function of s . The weighting functions are chosen to retain that part of the integrand containing $\ln(1/|\varrho_0|)$ and to eliminate that part of the integrand containing θ in the contour along the j -axis. The evaluations of the Taylor coefficients at various transmission zero locations follow.

Zero of t at infinity.--The form of the Taylor series expansion of $\ln(1/\varrho_0)$ about infinity is (from Appendix B)

$$\ln \frac{1}{\varrho_0} = F_1 \frac{1}{s} + F_3 \frac{1}{s^3} + \dots + F_{2m-1} \frac{1}{s^{2m-1}} + \dots, \quad (D-2)$$

where

$$F_{2m-1} = \frac{1}{2m-1} \left(\sum_1^n S_{ok}^{2m-1} - \sum_1^n S_{pi}^{2m-1} \right), \quad (D-3)$$

and $m=1, 2, 3, \dots$.

To obtain the value of F_1^∞ in terms of $\ln(1/|e_0|)$, $\ln(1/e_0)$ is multiplied by unity, and the resultant function is integrated around the contour shown in Fig. 1-D. Thus,

$$\oint \ln \frac{1}{e_0} ds = j \int_{-R}^R (\ln \frac{1}{|e_0|} + j\theta_0) d\omega + \quad (D-4)$$

$$\oint_{IV} \frac{F_1^\infty}{s} ds = 0.$$

The radius R is allowed to become infinite. Since no new singularities are encountered in this process, there results

$$\int_0^\infty \ln \frac{1}{|e_0|} d\omega = \frac{\pi}{2} F_1^\infty. \quad (D-5)$$

The contour of Fig. 1-D serves for the evaluation of F_3^∞ . The weighting function is s^2 , which brings the F_3^∞ coefficient into prominence and yields an integral relationship of

$$\oint s^2 \ln \frac{1}{e_0} ds = -j \int_{-R}^R \omega^2 (\ln \frac{1}{|e_0|} + j\theta) d\omega + \quad (D-6)$$

$$\oint_{IV} \frac{F_3^\infty}{s} ds = 0.$$

As before, as R becomes infinite,

$$\int_0^\infty \omega^2 \ln \frac{1}{|e_0|} d\omega = \frac{\pi}{2} F_3^\infty. \quad (D-7)$$

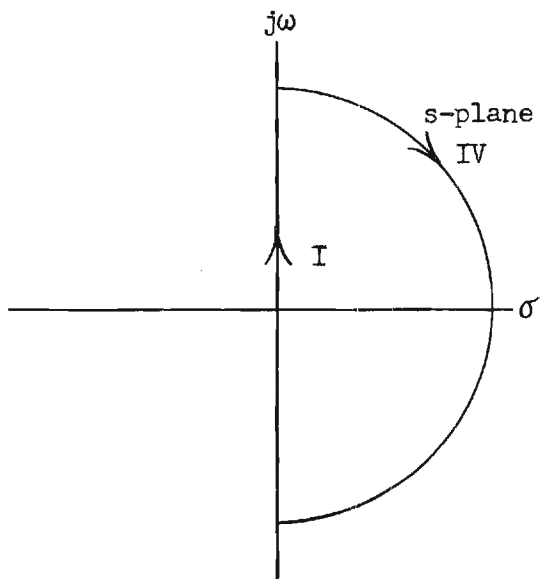


Fig. 1-D. Integration Contour

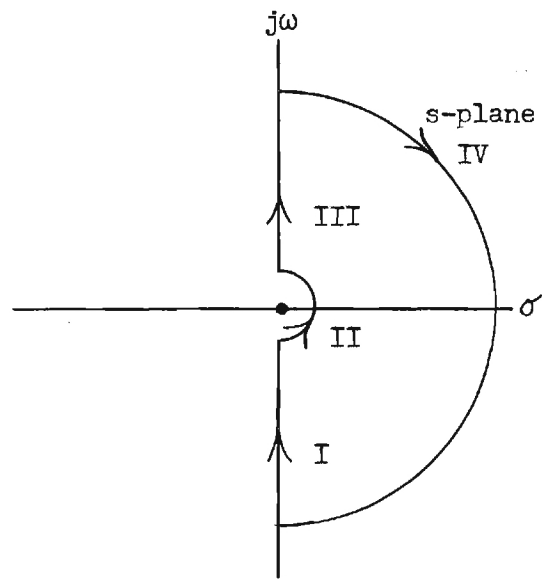


Fig. 2-D. Integration Contour

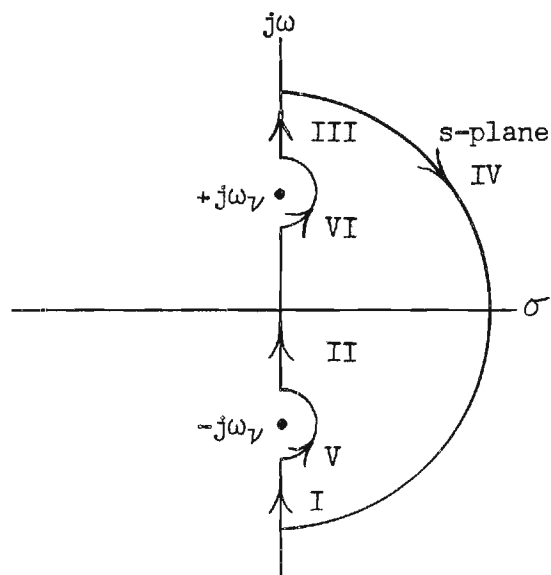


Fig. 3-D. Integration Contour

The process can be repeated to yield a general term of the form

$$\int_0^{\infty} \omega^{2m-2} \ln \frac{1}{|\rho_0|} d\omega = (-1)^{m+1} \frac{\pi}{2} F_{2m-1}^{\infty}, \quad (D-8)$$

where $m=1, 2, 3, \dots$.

Zero of t at the origin.--The form of the Taylor series expansion of $\ln(1/\rho_0)$ about zero is

$$\ln = F_1^0 s + F_3^0 s^3 + \dots + F_{2m-1}^0 s^{2m-1} + \quad (D-9)$$

where

$$F_{2m-1}^0 = \frac{1}{2m-1} \left(\sum_1^n \frac{-(-2m-1)}{S_{0k}} - \sum_1^n \frac{-(-2m-1)}{S_{pi}} \right), \quad (D-10)$$

and $m=1, 2, 3, \dots$.

The weighting function is $1/s^{2m}$, and the contour for integration is shown in Fig. 2-D. A semicircular indentation is made at the origin to avoid the pole of $[(1/s^{2m})\ln(1/\rho_0)]$ there. As a result of the contour integration, the general term is

$$\int_0^{\infty} \frac{1}{\omega^{2m}} \ln \frac{1}{|\rho_0|} d\omega = (-1)^{m+1} \frac{\pi}{2} F_{2m-1}^0, \quad (D-11)$$

where $m=1, 2, 3, \dots$.

Zero of t at $j\omega_v$.--When a transmission zero occurs at $j\omega_v$, a point on the real frequency axis, the Taylor series for $\ln(1/\rho_0)$ about $j\omega_v$ is

$$\ln 1/\rho_0 = jG_0^{\omega_v} + F_1^{\omega_v}(s-j\omega_v) + jG_2^{\omega_v}(s-j\omega_v)^2 + \text{(D-12)}$$

$$F_3^{\omega_v}(s-j\omega_v)^3 + jG_4^{\omega_v}(s-j\omega_v)^4 + \dots$$

$$\dots + jG_{2m-2}^{\omega_v}(s-j\omega_v)^{2m-2} + F_{2m-1}^{\omega_v}(s-j\omega_v)^{2m-1},$$

where

$$jG_0^{\omega_v} = \left(\ln \frac{1}{\rho_0} \right)_{s=j\omega_v}, \quad \text{(D-13)}$$

$$jG_{2m-2}^{\omega_v} = \frac{1}{2m-2} \left[\sum_1^n (s_{0k} - j\omega_v)^{-(2m-2)} - \sum_1^n (s_{pi} - j\omega_v)^{-(2m-2)} \right], \quad \text{(D-14)}$$

$$F_{2m-1}^{\omega_v} = \frac{1}{2m-2} \left[\sum_1^n (s_{0k} - j\omega_v)^{-(2m-1)} - \sum_1^n (s_{pi} - j\omega_v)^{-(2m-1)} \right], \quad \text{(D-15)}$$

and $m=1, 2, 3, \dots$.

The weighting function for obtaining the F^{ω_v} 's is

$$\left(\frac{1}{(s-j\omega_v)^{2m}} + \frac{1}{(s+j\omega_v)^{2m}} \right) = \frac{(s+j\omega_v)^{2m} + (s-j\omega_v)^{2m}}{(s^2 + \omega_v^2)^{2m}}, \quad \text{(D-16)}$$

where $m=1, 2, 3, \dots$. Use of the contour of Fig. 3-D leads to

$$\int_0^\infty \frac{(\omega + \omega_v)^{2m} + (\omega - \omega_v)^{2m}}{2(\omega_v^2 - \omega^2)^{2m}} \ln \frac{1}{|\rho_0|} d\omega = (-1)^{m+1} \frac{\pi}{2} F_{2m-2}^{\omega_v}, \quad \text{(D-17)}$$

where $m=1, 2, 3, \dots$.

The weighting function for obtaining the G^{ω_v} 's is

$$\left(\frac{1}{(s-j\omega_v)^{2m-1}} - \frac{1}{(s+j\omega_v)^{2m-1}} \right) = \frac{(s+j\omega_v)^{2m-1} - (s-j\omega_v)^{2m-1}}{(s^2 - \omega_v^2)^{2m-1}}, \quad (D-18)$$

where $m=1, 2, 3, \dots$. Use of the contour of Fig. 3-D leads to

$$\int_0^\infty \frac{(\omega + \omega_v)^{2m-1} - (\omega - \omega_v)^{2m-1}}{2(\omega_v^2 - \omega^2)^{2m-1}} \ln \frac{1}{|e_0|} d\omega = (-1)^m \frac{\pi}{2} G_{2m-2}^{\omega_v}, \quad (D-19)$$

where $m=1, 2, 3, \dots$.

Zero of t at σ_v .—When a transmission zero occurs at σ_v , a point on the positive real axis, the Taylor series expansion for $\ln(1/e_0)$ about σ_v is

$$\ln 1/e_0 = F_0^{\sigma_v} + F_1^{\sigma_v}(s - \sigma_v) + \dots + F_{m-1}^{\sigma_v}(s - \sigma_v)^{m-1} + \dots, \quad (D-20)$$

where

$$F_0^{\sigma_v} = \left(\ln \frac{1}{e_0} \right)_{s=\sigma_v}, \quad (D-21)$$

$$F_m = \frac{1}{m!} \left[\sum_{k=1}^n (s_{0k} - \sigma_v)^{-m} - \sum_{i=1}^n (s_{pi} - \sigma_v)^{-m} \right], \quad (D-22)$$

and $m=1, 2, 3, \dots$.

The weighting function is

$$\frac{1}{(s - \sigma_v)^m} + (-1)^m \frac{1}{(s + \sigma_v)^m}. \quad (D-23)$$

The contour of Fig. 1-D can be used since there are no singularities of the integrand along the imaginary axis. The general term is found to be

$$\int_0^\infty \frac{(\sigma_v + j\omega)^m + (\sigma_v - j\omega)^m}{2(\sigma_v^2 + \omega^2)^m} \ln \frac{1}{|\rho_0|} d\omega = (-1)^{m+1} \frac{\pi}{2} F_{m-1}^{s_v}, \quad (D-24)$$

where $m=1, 2, 3, \dots$.

Zero of t at $s_v = \sigma_v + j\omega_v$.—When a transmission zero occurs at $s_v = \sigma_v + j\omega_v$, a point in the right-half s -plane (not on the positive real axis), the Taylor series expansion for $\ln(1/\rho_0)$ about s_v is

$$\begin{aligned} \ln 1/\rho_0 = & (F_0^{s_v} + jG_0^{s_v}) + (F_1^{s_v} + jG_1^{s_v})(s - s_v) + \quad (D-25) \\ & (F_2^{s_v} + jG_2^{s_v})(s - s_v)^2 + \dots + \\ & (F_{2m-2}^{s_v} + jG_{2m-2}^{s_v})(s - s_v)^{2m-2} + (F_{2m-1}^{s_v} + jG_{2m-1}^{s_v})(s - s_v)^{2m-1} + \dots \end{aligned}$$

where

$$(F_0^{s_v} + jG_0^{s_v}) = \left(\ln \frac{1}{|\rho_0|} \right)_{s=s_v}, \quad (D-26)$$

$$(F_m^{s_v} + jG_m^{s_v}) = \frac{1}{m} \left[\sum_k^n (s_{ok} - s_v)^{-m} - \sum_i^n (s_{pi} - s_v)^{-m} \right], \quad (D-27)$$

and $m=1, 2, 3, \dots$.

The procedure for obtaining the F^{s_v} 's and G^{s_v} 's in terms of $\ln(1/|\rho_0|)$ is the same as that used for obtaining the corresponding terms of other Taylor series expansions. However, care must be exercised in choosing the weighting function. The weighting function is of the form

$$\frac{a}{(s - s_v)^m} + \frac{b}{(s - \bar{s}_v)^m} + \frac{c}{(s + s_v)^m} + \frac{d}{(s + \bar{s}_v)^m}, \quad (D-28)$$

where a , b , c , and d are real numbers. To account for the even real terms, i.e., F_2^{sv} , F_4^{sv} , etc., it is apparent that a choice of $a=1$, $c=1$, $b=-1$, and $d=-1$ with m odd is a consistent choice for expression (D-28). Similarly, a multiplying factor for obtaining the even imaginary terms, i.e., G_2^{sv} , G_4^{sv} , etc., has $a=1$, $c=-1$, $b=-1$, and $d=1$ with m odd. The odd real terms, i.e., F_1^{sv} , F_3^{sv} , etc., can be found using $a=1$, $c=1$, $b=1$, and $d=1$ with m even. The odd imaginary terms, i.e., G_1^{sv} , G_3^{sv} , etc., can be found using $a=1$, $c=-1$, $b=1$, and $d=-1$ with m even.

By using the appropriate weighting function and the contour of Fig. 1-D, the following general terms are found to be

$$\int_0^\infty \frac{1}{4} \left[\frac{1}{(j\omega - s_v)^{2m-1}} - \frac{1}{(j\omega - \bar{s}_v)^{2m-1}} + \frac{1}{(j\omega + s_v)^{2m-1}} - \right. \quad (D-29)$$

$$\left. \frac{1}{(j\omega + \bar{s}_v)^{2m-1}} \right] \ln \frac{1}{|Q_0|} d\omega = -\frac{\pi}{2} F_{2m-2},$$

$$\int_0^\infty \frac{1}{4} \left[\frac{1}{(j\omega - s_v)^{2m-1}} - \frac{1}{(j\omega - \bar{s}_v)^{2m-1}} - \frac{1}{(j\omega + s_v)^{2m-1}} + \right. \quad (D-30)$$

$$\left. \frac{1}{(j\omega + \bar{s}_v)^{2m-1}} \right] \ln \frac{1}{|Q_0|} d\omega = -\frac{\pi}{2} j G_{2m-2},$$

$$\int_0^\infty \frac{1}{4} \left[\frac{1}{(j\omega - s_v)^{2m}} + \frac{1}{(j\omega - \bar{s}_v)^{2m}} + \frac{1}{(j\omega + s_v)^{2m}} + \right. \quad (D-31)$$

$$\left. \frac{1}{(j\omega + \bar{s}_v)^{2m}} \right] \ln \frac{1}{|Q_0|} d\omega = -\frac{\pi}{2} F_{2m-1},$$

$$\int_0^{\infty} \frac{1}{4} \left[\frac{1}{(j\omega - s_r)^{2m}} + \frac{1}{(j\omega - \bar{s}_r)^{2m}} - \frac{1}{(j\omega + s_r)^{2m}} - \frac{1}{(j\omega + \bar{s}_r)^{2m}} \right] \ln \frac{1}{|\rho_0|} d\omega = -\frac{\pi}{2} j G_{2m-1}, \quad (D-32)$$

where $m=1, 2, 3, \dots$.

Contour Integral Values for ρ_0 and Right-half s-plane Reflection Zeros

The reflection coefficient ρ_0 and the corresponding return loss $\ln(1/|\rho_0|)$ have no singularities in the right-half s-plane. However, any discussion of the contour integrals is not complete unless right-half s-plane zeros of reflection are considered. In this study the addition of right-half s-plane zeros of reflection do not have any effect on the magnitude of a reflection coefficient at real frequencies. When right-half s-plane reflection zeros are intentionally used, the addition is accomplished by placing a left-half s-plane pole of reflection diametrically opposite the added right-half s-plane zero. This cancels out the effect of the right-half s-plane zero as far as magnitudes at real frequencies are concerned. Thus, a general reflection coefficient such as ρ (or ρ_1 or ρ_2) can appear in the form

$$\rho = (\pm 1) \rho_0 \frac{\prod_{j=1}^k (s - s_{rj})}{\prod_{j=1}^k (s + s_{rj})}, \quad (D-33)$$

where ρ_0 is that part of ρ which has no right-half s-plane zeros or poles. The s_{rj} 's have, of course, positive real parts. The (± 1) is a multiplier for obtaining the proper sign of ρ in case it is necessary to adjust the sign to make ρ_0 non-negative at s equal to zero.

The return loss $\ln(1/\rho)$ is either analytic at zeros of transmission at the outset or can be modified suitably if a zero of reflection and a zero of transmission should occur simultaneously in the right-half s -plane. (See Appendix C.) Thus, $\ln(1/\rho)$ can be expanded in a Taylor series about the transmission zeros. In the following section the coefficients of the Taylor series for $\ln(1/\rho)$ are related to the coefficients of $\ln(1/\rho_0)$ and to the integral relationships developed for $\ln(1/|\rho_0|)$. In this regard, it is pointed out again that $|\rho_0|$ and $|\rho|$ are equal on the imaginary axis (at real frequencies), and $|\rho|$ can be interchanged with $|\rho_0|$ at will at real frequencies. The descriptions of the relationships between the expansions for $\ln(1/\rho_0)$ and $\ln(1/\rho)$ follow.

Zero of t at infinity.--The general coefficient of the Taylor series expansion of $\ln(1/\rho)$ about infinity is taken from equation (B-19) as

$$A_{2m-1}^{\infty} = \frac{1}{2m-1} \left(\sum_1^n s_{0i}^{2m-1} - \sum_1^n s_{pi}^{2m-1} \right), \quad (D-34)$$

where $m=1, 2, 3, \dots$. The s_{0i} 's of equation (D-34) are zeros of ρ located anywhere in the s -plane. If of the total of n zeros of ρ there are k right-half s -plane zeros, equation (D-34) can be rearranged as

$$A_{2m-1}^{\infty} = \frac{1}{2m-1} \left[\left(\sum_1^{n-k} s_{0i}^{2m-1} - \sum_1^k s_{rj}^{2m-1} - \sum_1^n s_{pi}^{2m-1} \right) + \left(\sum_1^k s_{rj}^{2m-1} - \sum_1^k s_{rj}^{2m-1} \right) \right]. \quad (D-35)$$

The expression in the first set of parentheses of equation (D-35) can be

recognized as stemming from ρ_0 and is, in fact, F_{2m-1}^∞ . Use of equation (D-8) and the fact that $|\rho_0|$ is equal to $|\rho|$ for real frequencies lead to

$$A_{2m-1}^\infty = (-1)^{m+1} \frac{2}{\pi} \int_0^\infty \omega^{2m-2} \ln \frac{1}{|\rho|} d\omega + \frac{2}{2m-1} \sum_1^k S_{rj}^{2m-1}, \quad (D-36)$$

$$(-1)^{m+1} \frac{\pi}{2} F_{2m-1}^\infty = \int_0^\infty \omega^{2m-2} \ln \frac{1}{|\rho|} d\omega = (-1)^{m+1} \frac{\pi}{2} \left(A_{2m-1}^\infty - \frac{2}{2m-1} \sum_1^k S_{rj}^{2m-1} \right), \quad (D-37)$$

where $m=1, 2, 3, \dots$.

Zero of t at the origin.---A manipulation identical with that above can be performed with respect to a zero of transmission at the origin to yield

$$A_{2m-1}^0 = (-1)^{m+1} \frac{2}{\pi} \int_0^\infty \frac{1}{\omega^{2m}} \ln \frac{1}{|\rho|} d\omega + \frac{2}{2m-1} \sum_1^k S_{rj}^{-(2m-1)}, \quad (D-38)$$

$$(-1)^{m+1} \frac{\pi}{2} F_{2m-1}^0 = \int_0^\infty \frac{1}{\omega^{2m}} \ln \frac{1}{|\rho|} d\omega = \quad (D-39)$$

$$(-1)^{m+1} \frac{\pi}{2} \left(A_{2m-1}^0 - \frac{2}{2m-1} \sum_1^k S_{rj}^{-(2m-1)} \right),$$

where $m=1, 2, 3, \dots$.

Zero of t at $j\omega_v$.---The Taylor series expansion of $\ln(1/\rho)$ about a zero of transmission on the real frequency axis has both real and imaginary coefficients. The expression for ρ given in equation (D-33) can be used to obtain

$$\ln \frac{1}{\rho} = \ln \frac{1}{\rho_0} + \ln \left[\frac{\prod_{j=1}^k (s + s_{rj})}{\prod_{j=1}^k (s - s_{rj})} \right] + \ln (\pm 1). \quad (D-40)$$

Thus, from equations (D-13) and (B-12),

$$jB_0^{\omega_v} = jG_0^{\omega_v} + \ln \left[\frac{\prod_{j=1}^k (s + s_{rj})}{\prod_{j=1}^k (s - s_{rj})} \right]_{s=j\omega_v}, \quad (D-41)$$

and from equation (D-19),

$$jB_0^{\omega_v} = -j \frac{2}{\pi} \int_0^\infty \frac{\omega_v}{(\omega_v^2 - \omega^2)} \ln \frac{1}{|\rho|} d\omega + \ln \left[\frac{\prod_{j=1}^k (s + s_{rj})}{\prod_{j=1}^k (s - s_{rj})} \right]_{s=j\omega_v}. \quad (D-42)$$

After suitable manipulation, the general terms are given by

$$(-1)^{m+1} \frac{\pi}{2} F_{2m-1}^{\omega_v} = \int_0^\infty \frac{(\omega + \omega_v)^{2m} + (\omega - \omega_v)^{2m}}{2(\omega_v^2 - \omega^2)^{2m}} \ln \frac{1}{|\rho|} d\omega = \quad (D-43)$$

$$(-1)^{m+1} \frac{\pi}{2} \left(A_{2m-1}^{\omega_v} - \frac{2}{2m-1} \operatorname{Re} \sum_{j=1}^k (s_{rj} - j\omega_v)^{-(2m-1)} \right),$$

$$A_{2m-1}^{\omega_v} = (-1)^{m+1} \frac{2}{\pi} \int_0^\infty \frac{(\omega + \omega_v)^{2m} + (\omega - \omega_v)^{2m}}{2(\omega_v^2 - \omega^2)^{2m}} \ln \frac{1}{|\rho|} d\omega + \quad (D-44)$$

$$\frac{2}{2m-1} \operatorname{Re} \sum_{j=1}^k (s_{rj} - j\omega_v)^{-(2m-1)},$$

where $m=1, 2, 3, \dots$, and Re means "real part of."

Similarly, the imaginary terms are given by

$$j(-1)^{m+1} \frac{\pi}{2} G_{2m} = j \int_0^{\infty} \frac{(\omega + \omega_v)^{2m+1} - (\omega - \omega_v)^{2m+1}}{2(\omega_v^2 - \omega^2)^{2m+1}} \ln \frac{1}{|\rho|} d\omega = \quad (D-45)$$

$$j(-1)^{m+1} \frac{\pi}{2} \left(B_{2m}^{\omega_v} - \frac{1}{m} \text{Im} \sum_1^k (s_{rj} - j\omega_v)^{-2m} \right),$$

$$j B_{2m}^{\omega_v} = j(-1)^{m+1} \frac{2}{\pi} \int_0^{\infty} \frac{(\omega + \omega_v)^{2m+1} - (\omega - \omega_v)^{2m+1}}{2(\omega_v^2 - \omega^2)^{2m+1}} \ln \frac{1}{|\rho|} d\omega + \quad (D-46)$$

$$\frac{1}{m} \text{Im} \sum_1^k (s_{rj} - j\omega_v)^{-2m},$$

where $m=1, 2, 3, \dots$, and Im means "imaginary part of."

Zero of t at σ_v . -- From equations (D-33), (B-16), and (D-21), it is evident that

$$A_o^{\sigma_v} = F_o^{\sigma_v} + \ln \left(\frac{\prod_1^k (s + s_{rj})}{\prod_1^k (s - s_{rj})} \right)_{s=\sigma_v}. \quad (D-47)$$

In addition, equations (D-22), (D-24), and (D-33) indicate that

$$(-1)^{m+1} \frac{\pi}{2} F_m^{\sigma_v} = \int_0^{\infty} \frac{(\sigma_v + j\omega)^{m+1} + (\sigma_v - j\omega)^{m+1}}{2(\sigma_v^2 + \omega^2)^{m+1}} \ln \frac{1}{\rho} d\omega = \quad (D-48)$$

$$(-1)^{m+1} \frac{\pi}{2} \left(A_m^{\sigma_v} - \frac{1}{m} \left[\sum_1^k (s_{rj} - \sigma_v)^{-m} - \sum_1^k (-s_{rj} - \sigma_v)^{-m} \right] \right),$$

$$A_m^{\sigma_v} = (-1)^{m+1} \frac{2}{\pi} \int_0^{\infty} \frac{(\sigma_v + j\omega)^{m+1} + (\sigma_v - j\omega)^{m+1}}{2(\sigma_v^2 + \omega^2)^{m+1}} \ln \frac{1}{|\rho|} d\omega + \quad (D-49)$$

$$\frac{1}{m} \left[\sum_1^k (s_{rj} - \sigma_v)^{-m} - \sum_1^k (-s_{rj} - \sigma_v)^{-m} \right],$$

where $m=1, 2, 3, \dots$.

Zero of t at $s_v = \sigma_v + j\omega_v$.---The coefficients for the Taylor series expansions of $\ln(1/\rho)$ and $\ln(1/\rho_0)$ about s_v are, in general, complex.

Application of the principles of the preceding sections leads to

$$A_0^{s_v} + j B_0^{s_v} = F_0^{s_v} + j G_0^{s_v} + \ln \left(\frac{\prod_{i=1}^k (s + s_{ri})}{\prod_{j=1}^k (s - s_{rj})} \right)_{s=s_v}, \quad (D-50)$$

$$A_m^{s_v} + j B_m^{s_v} = F_m^{s_v} + j G_m^{s_v} + \frac{1}{m} \left[\sum_{j=1}^k (s_{rj} - s_v)^{-m} - \sum_{j=1}^k (-s_{rj} - s_v)^{-m} \right], \quad (D-51)$$

where $m=1, 2, 3, \dots$.

Concluding remarks on contour integrals.---The A 's and B 's developed above are the terms which would result if the contour integration were made with $\ln(1/\rho)$ and weighting functions instead of first getting the F 's and G 's from application of $\ln(1/\rho_0)$. In the reflection coefficient functions of this study, the left-half s -plane zeros of reflection are obtained from the Tchebycheff approximating function while the right-half s -plane zeros of reflection are quantities to be added to yield consistent simultaneous equations or to adjust the tolerance of transmission. Thus, it is seen that the use of $\ln(1/\rho_0)$ expansions is justified since the zeros and poles of ρ_0 are predetermined by the chosen Tchebycheff approximating function.

APPENDIX E

EXAMPLES

Sample Calculation, No Added Right-half s-plane Reflection Zeros

Specification of data.--A particularly simple matching problem occurs when the source and load networks are equal capacitances and thus have transmission zeros at infinity. For illustrative purposes, the data of the problem state that $n=3$ and $\omega_c=1$. A sketch of the network arrangement is given in Fig. 1-E.

Calculation of $A_1^{(1)}$ and $A_1^{(2)}$.--The $A_1^{(1)}$ term for the entire system is identical with the $A_1^{(1)}$ term derived from the load network. In the same way, the $A_1^{(2)}$ term for the entire system is identical with the $A_1^{(2)}$ term derived from the source network. Sketches of circuits for obtaining $A_1^{(1)}$ and $A_1^{(2)}$ are shown in Fig. 2-E. In this case, ρ_1' and ρ_2''' are the same, and a calculation for one suffices for the other. Reference to Fig. 2-E indicates that

$$Z_1' = \frac{1}{1 + s}, \quad (\text{E-1})$$

from which

$$\rho_1' = \frac{-s}{s + 2} \quad (\text{E-2})$$

The methods of Appendix B indicate that $A_1^{(1)} = A_1^{(2)} = 2$. It follows that $A_1^{(1)} = A_1^{(2)} = 2$.

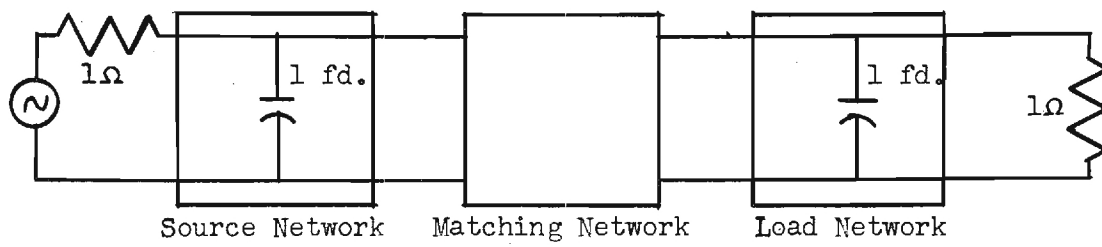
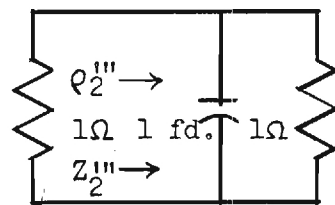
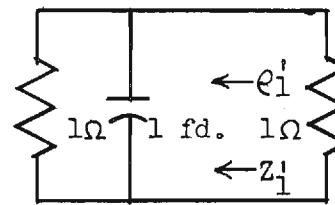


Fig. 1-E. Simple System of Networks



(a) Source Network



(b) Load Network

Fig. 2-E. Circuits for Obtaining Network Parameters

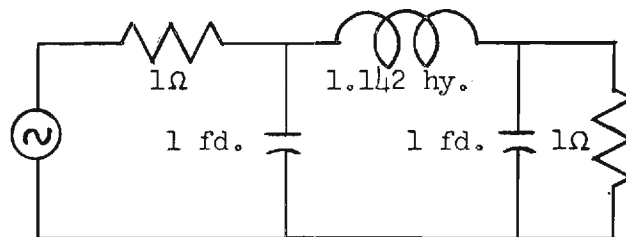


Fig. 3-E. Complete Network

Calculation of hyperbolic functions of a and b.--From equations (106)

and (109) with $\sigma=0$, there results

$$Z = \frac{\sinh a - \sinh b}{\sin \pi/6} + 2 \sum_1^k s_{rj} , \quad (\text{E-3})$$

$$Z = \frac{\sinh a + \sinh b}{\sin \pi/6} + 2 \sum_1^k s_{rj} . \quad (\text{E-4})$$

Since the greatest possibility for gain occurs in the absence of right-half s-plane zeros of reflection, the summations of the s_{rj} 's in equations (E-3) and (E-4) are set equal to zero. When a simultaneous solution is made, the results are $a=0.8814$, $b=0.0000$, $\sinh a=1.0000$, $\cosh a=1.4142$, $\sinh b=0.0000$, and $\cosh b=1.0000$.

Calculation of ρ_1 and Z_1 .--The pole and zero locations are found from equations (100) and (101) as

$$\left. \begin{aligned} s_{p1} &= -0.5000 + j1.2530 \\ s_{p2} &= -1.0000 + j0.0000 \\ s_{p3} &= -0.5000 - j1.2530 \end{aligned} \right\} , \quad (\text{E-5})$$

$$\left. \begin{aligned} s_{o1} &= 0.0000 + j0.8660 \\ s_{o2} &= 0.0000 + j0.0000 \\ s_{o3} &= 0.0000 - j0.8660 \end{aligned} \right\} . \quad (\text{E-6})$$

When the poles and zeros are arranged to form the denominator and numerator of ρ_1 , the result is

$$\rho_1 = h \frac{s^3 + 0.75s}{s^3 + 2s^2 + 2.75s + 1.75} . \quad (\text{E-7})$$

The value of $\rho_1' = -1$ at $s = \infty$. In order for ρ_1 to be equal to ρ_1' at infinity, the value of $h = -1$ in equation (E-7). The value of Z_1 is found by equation (148) as

$$Z_1(s) = \frac{2s^2 + 2s + 1.75}{2s^3 + 2s^2 + 3.5s + 1.75}, \quad (\text{E-8})$$

from which

$$Y_1(s) = \frac{2s^3 + 2s^2 + 3.5s + 1.75}{2s^2 + 2s + 1.75}. \quad (\text{E-9})$$

Determination of the network values.—Since the form of the complete network and the values of the two end elements are known, it is possible to determine the matching network values either by starting at one end and extracting elements of a ladder network until the other end is reached or by starting at each end with the extraction process and working toward the middle. The first method is used here.

The complete network must start with a shunt capacitance of one farad. (This is the first element of the known load network.) The continued fraction technique for obtaining the ladder form of networks yields

$$Y_1(s) = s + \frac{1}{1.142s + \frac{1}{s + \frac{1}{1}}}, \quad (\text{E-10})$$

which indicates that the second element of the complete system or the first element of the matching network is a series inductance of 1.142

henrys. Since $n=3$, the 1.142 henry inductance is the only element of the matching section. The complete network is shown in Fig. 3-E. The upper and lower limits of $|Q_1|$ in the pass band are 0.141 and zero, respectively.

Sample Calculation, Added Simple Right-half s-plane Reflection Zero
Specification of data.--In the next example, the source network consists of a one henry series inductance, while the load network consists of a shunt capacitance of one farad and a series inductance of two henrys. The data of the problem state that $n=4$ and $\omega_c=1$. A sketch of the network arrangement is shown in Fig. 4-E.

Calculation of the A^∞ terms.--In this example, two A^∞ terms are found from the load network, and one A^∞ term is found from the source network. The methods of Appendix B yield values of $A_1^\infty=2$, $A_3^\infty=-1/3$, and $A_1^\infty=2$.

Calculation of hyperbolic functions of a and b and calculation of $\xi_{s,j}$.--Reference to equations (106), (107), and (109) indicates that a simultaneous solution of the gain integral equations for this example cannot be obtained unless added right-half s-plane reflection zeros are introduced. If it is assumed that there are to be no added double reflection zeros (i.e., $\nu=0$) and that a simple zero on the positive axis is used, equations (106), (107), and (109) reduce to

$$Z = \frac{\sinh a - \sinh b}{\sin \pi/8} + 2\mathcal{E}, \quad (\text{E-11})$$

$$-\frac{1}{3} = \frac{1}{4} \left[\frac{\sinh 3a - \sinh 3b}{3 \sin 3\pi/8} + \frac{\sinh a - \sinh b}{\sin \pi/8} \right] + \frac{2\mathcal{E}^3}{3}, \quad (\text{E-12})$$

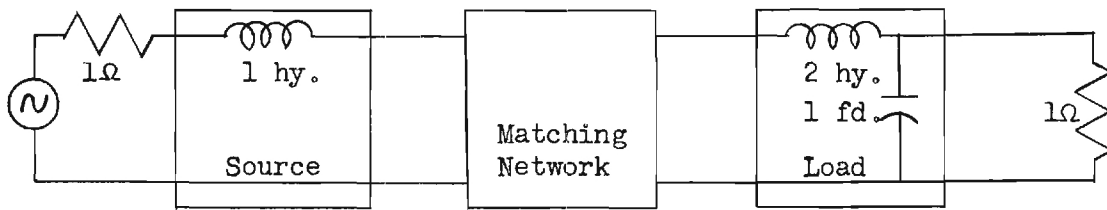


Fig. 4-E. Simple System of Networks

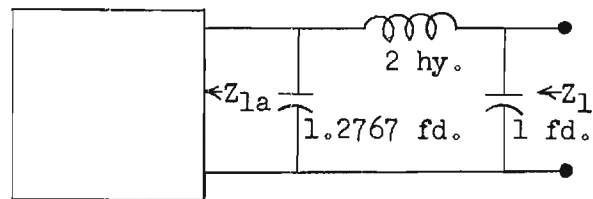


Fig. 5-E. Partially Reduced Network

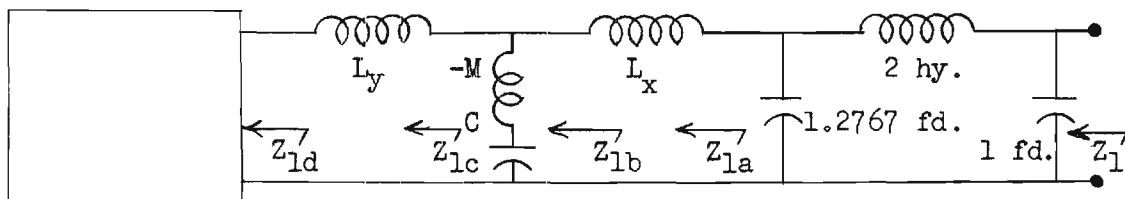


Fig. 6-E. Network with Type C Section

$$Z = \frac{\sinh a + \sinh b}{\sin \pi/8} + 2\mathcal{S}, \quad (\text{E-13})$$

where \mathcal{S} is a point on the positive real axis.

A simultaneous solution of equations (E-11) through (E-13) yields $a=0.3974$, $b=0.0000$, $\mathcal{S}=0.4671$, $\sinh a=0.4079$, $\sinh 3a=1.4950$, $\cosh a=1.0800$, $\cosh 3a=1.7986$, $\sinh b=0.0000$, $\sinh 3b=0.0000$, $\cosh b=1.0000$, and $\cosh 3b=1.0000$.

Calculation of ρ_1 and Z_1 .--The location of the Tchebycheff poles and zeros of ρ_1 are found from equations (100) and (101) as

$$\left. \begin{aligned} S_{p1} &= 0.0000 + j0.9239 \\ S_{p2} &= 0.0000 + j0.3827 \\ S_{p3} &= 0.0000 - j0.3827 \\ S_{p4} &= 0.0000 - j0.9239 \end{aligned} \right\}, \quad (\text{E-14})$$

$$\left. \begin{aligned} S_{o1} &= -0.1561 + j0.9978 \\ S_{o2} &= -0.3768 + j0.4133 \\ S_{o3} &= -0.3768 - j0.4133 \\ S_{o4} &= -0.1561 - j0.9978 \end{aligned} \right\}. \quad (\text{E-15})$$

A denominator formed from the poles of equations (E-14) and a numerator formed from the zeros of equations (E-15) would result in a reflection coefficient which has the proper magnitude at real frequencies. However, the effect of the added zero of reflection at $s=0.4671$ must be accounted for. The factors $[(s-0.4671)(s+0.4671)]$ appear in the numerator of ρ_1 , and the factors $[(s+0.4671)(s+0.4671)]$ appear in the denominator of ρ_1 . The above factors are used in conjunction with factors formed from the zeros and poles of equations (E-14) and (E-15)

to obtain

$$\rho_1 = h \frac{(s^6 + 0.7818s^4 - 0.0932s^2 + 0.0273)}{(s^6 + 2s^5 + 2.7818s^4 + 2.5636s^3 + 1.4704s^2 + 0.4870s + 0.0696)} \quad (\text{E-16})$$

Since the element of the source network is a shunt capacitance and $\rho_1=1$ at $s=\infty$, $h=-1$ in equation (E-16). The expression for Z_1 is

$$Z_1 = \frac{(2s^5 + 2s^4 + 2.5636s^3 + 1.5636s^2 + 0.4870s + 0.0969)}{(2s^6 + 2s^5 + 3.5636s^4 + 2.5636s^3 + 1.3772s^2 + 0.4870s + 0.0423)} \quad (\text{E-17})$$

Determination of network values.--The continued fraction technique can be used to extract successively the shunt capacitance and the series inductance which make up the load network. The network remaining after the load network is removed contains elements which contribute to two zeros of transmission at infinity and one zero of transmission at $s=j$. One of the zeros at infinity must be reserved for the source network, while the other zero at infinity and the zero at j are contained within the matching network.

It is immaterial which zero of transmission in the matching network is considered first. For illustrative purposes, the element leading to a zero of transmission at infinity is removed first. After the elements of the load network are extracted, the continued fraction technique indicates that the first element of the matching network is a shunt capacitance equal to 1.2767 farads. When the capacitance is removed, the remaining impedance Z_{1a} is given by

$$Z_{1a} = \frac{(0.7833s^3 + 0.7833s^2 + 0.4024s + 0.0969)}{(0.3765s^2 + 0.2665s + 0.0423)} \quad (E-18)$$

A sketch of the network at this stage is shown in Fig. 5-E.

A Darlington type C section is used to produce the transmission zero at s . The position of the type C section is shown in Fig. 6-E. At $s=s$ (in this case, 0.4672), the shunt M-C branch reduces to a short circuit. Thus, $Z_{1a}(s)=2.1514$, which is also the impedance of L_x at $s=0.4672$. An L_x of 4.6057 henrys results. When L_x is removed from Z_{1a} , the expression is

$$Z_{1b} = \frac{(-0.9506s^3 - 0.4440s^2 + 0.2074s + 0.0969)}{(0.3765s^2 + 0.2665s + 0.0423)} \quad (E-19)$$

At $s=0.4672$, Y_{1b} (the reciprocal of Z_{1b}) equals the admittance of the M-C branch, which is

$$Y_{M-C} = \left(\frac{-s/M}{s^2 - 1/MC} \right), \quad (E-20)$$

where $1/MC=s^2$ (or $[0.4672]^2$) (43). The value of M is found to be 1.6663 henrys, and that of C is 2.7505 farads. By using these values, it is seen that

$$Y_{M-C} = \left(\frac{0.6001}{s^2 - 0.2182} \right). \quad (E-21)$$

The admittance of equation (E-21) can be subtracted from Y_{1b} to yield Y_{1c} . The corresponding Z_{1c} is given by

$$Z_{lc} = 4.8994s + 2.2886. \quad (E-22)$$

It is seen that Z_{lc} is a series combination of a 4.8994 henry inductance and a 2.2886 ohm resistance.

Through use of close-coupled relationships of the Darlington type C section shown in Fig. 15, it is found that $L_y = 2.6108$ henrys, $L_1 = 0.9445$ henrys, and $L_2 = 2.9394$ henrys. When the complex impedance associated with L_y is subtracted from Z_{lc} , Z_{ld} becomes

$$Z_{ld} = 2.2886s + 2.2886, \quad (E-23)$$

which represents a 2.2886 henry inductance in series with a 2.2886 ohm resistance. In order for a one-ohm resistance to be used as the generator resistance, an ideal transformer of turns ratio 1:1.5128 must be inserted at some point between the one-ohm resistance and the right-hand side of the network. By inserting the transformer between L_y and Z_{ld} , the source network is reconstructed. The final steps leading to a complete system are shown in Fig. 7-E. The upper and lower limits of $|Q_1|$ in the pass band are 0.5119 and zero, respectively.

Sample Calculation, Added Double Right-half s-plane Reflection Zero

General.--When the source and load networks have the same (or numerically equal) first elements, the lower limit of $|Q_1|$ is zero unless double reflection zeros are used. It is recalled that a zero lower limit of $|Q_1|$ may not be desirable from a matching standpoint. Double right-half s-plane reflection zeros are available to raise the lower limit of $|Q_1|$. As is pointed out in the text, this process reduces the ripple but also reduces the gain. A compromise between gain and tolerance must be

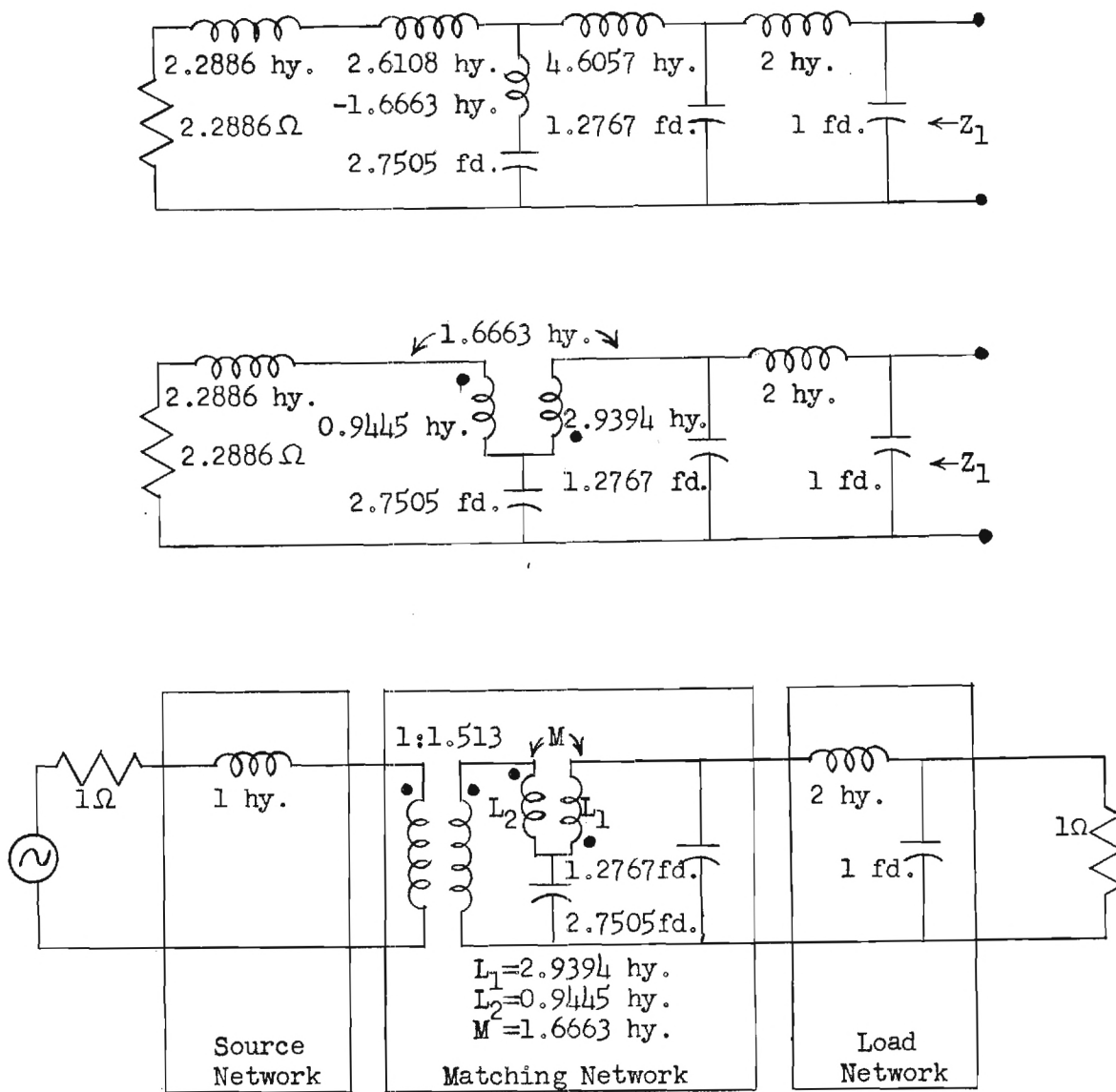


Fig. 7-E. Development of Final Network

accepted when double zeros of reflection are added. The sample calculation of this section uses the source and load networks of the first example in order to show comparative data.

Specification of data.--The network arrangement of Fig. 1-E shows the element values for the load and source networks. As before, $A_1^{(1)}=2$, $A_1^{(2)}=2$, $n=3$, and $\omega_c=1$. In addition, σ , the added double zero point, is chosen 0.25.

Calculation of hyperbolic functions of a and b.--When $\sigma=0.25$, equations (106) and (109) reduce to

$$1 = \frac{\sinh a - \sinh b}{\sin \pi/6}, \quad (\text{E-24})$$

$$2 = \frac{\sinh a + \sinh b}{\sin \pi/6}. \quad (\text{E-25})$$

A simultaneous solution yields $a=0.6931$, $b=0.2475$, $\sinh a=0.7500$, $\cosh a=1.2500$, $\sinh b=0.2500$, and $\cosh b=1.3078$.

Calculation of ρ_1 and Z_1 .--The Tchebycheff pole and zero locations of ρ_1 are given by

$$\left. \begin{aligned} S_{p1} &= -0.3750 + j1.0825 \\ S_{p2} &= -0.7500 + j0.0000 \\ S_{p3} &= -0.3750 - j1.0825 \end{aligned} \right\}, \quad (\text{E-26})$$

$$\left. \begin{aligned} S_{o1} &= -0.1250 + j0.8927 \\ S_{o2} &= -0.2500 + j0.0000 \\ S_{o3} &= -0.1250 - j0.8927 \end{aligned} \right\}. \quad (\text{E-27})$$

The factors $[(s-0.25)(s-0.25)]$ are incorporated into the numerator of

ρ_1 , and the factors $[(s+0.25)(s+0.25)]$ are incorporated into the denominator of ρ_1 . The values of ρ_1 and the corresponding Z_1 are

$$\rho_1 = -\frac{(s^5 + 0.6875s^3 - 0.2031s^2 - 0.0469s + 0.0127)}{(s^5 + 2s^4 + 2.6875s^3 + 2.0156s^2 + 0.6094s + 0.0615)}, \quad (\text{E-28})$$

$$Z_1 = \frac{(2s^4 + 2s^3 + 2.2187s^2 + 0.6563s + 0.0488)}{(2s^5 + 2s^4 + 3.3750s^3 + 1.8125s^2 + 0.5625s + 0.0742)}. \quad (\text{E-29})$$

Determination of network values.—The technique of obtaining the circuit values is much the same as that employed in the second example. After the one farad capacitance of the load network is removed, the remaining network contains two zeros of transmission at infinity and one zero of transmission at $s=0.25$. One of the transmission zeros at infinity is caused by the source network, while the other transmission zero at infinity and the transmission zero at 0.25 are caused by elements in the matching network. Hence, the matching network is made up of a series inductance and a Darlington type C section. It is immaterial which is removed first as far as the realizability of the network is concerned, although different orders of removal result in different element values. If the series inductance is removed first, the elements of the network can be evaluated as shown in Fig. 8-E. The upper and lower limits of $|\rho_1|$ in the pass band are 0.3173 and 0.2065, respectively. Reference to the results of the first example indicates the extent to which the limits of $|\rho_1|$ are changed by the double zero of reflection.

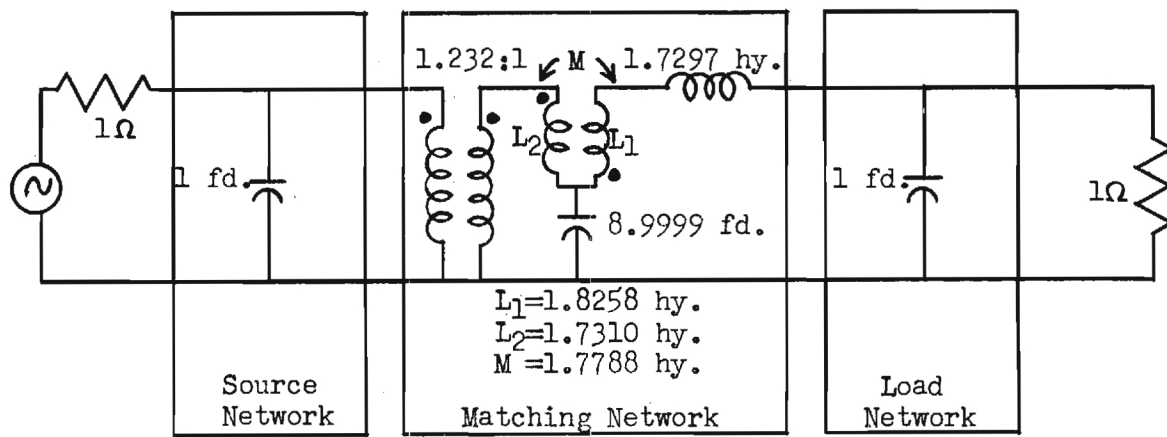


Fig. 8-E. Final Network

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